Dismantlable lattices in the mirror

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Abstract. We investigate properties which hold for both the lattice of a binary relation and for its 'mirror lattice', which is the lattice of the complement relation.

We first prove that the relations whose lattice is dismantlable correspond to the class of chordal bipartite graphs; we provide algorithmic tools to find a doubly irreducible element in such a lattice.

We go on to show that a lattice is dismantlable and its mirror lattice is also dismantlable if and only if both these lattices are planar.

Keywords: dismantlable lattice; planar lattice; mirror relation; chordal bipartite graph.

1 Introduction

A binary relation is associated with a bipartite graph and a concept (or Galois) lattice. Its complement relation, which we call the *mirror relation*, is associated with the corresponding *mirror bipartite graph* and *mirror lattice*.

This mirror lattice was investigated by *e.g.* Deiters and Erné [10], who examined the succession of lattices one can obtain by repeatedly computing the mirror relation, reducing it, and computing the mirror of the obtained relation.

Our area of interest is to find properties which are preserved in the mirror lattice. In [5], we extended the well-known property that a lattice which is a chain has a mirror lattice which is a chain: we showed that a lattice has an articulation point if and only if its mirror lattice has an articulation point (*i.e.* an element which is comparable to all the other elements, but is not extremum).

In this paper, we investigate dismantlable lattices. A lattice is said to be dismantlable if one can repeatedly remove a doubly irreducible element until the lattice becomes a chain. This class was investigated by several authors: Baker, Fishburn and Roberts [3] showed that all planar lattices are dismantlable; Rival [20] showed that removing a doubly irreducible element always defines a sublattice; Rival and Kelly [21] characterized dismantlable lattices as being 'crown-free'; recently, Brucker and Gély [8] studied co-atomistic dismantlable lattices.

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We will show that there is a strong mirror relationship between planar lattices and dismantlable lattices: a lattice and its mirror lattice are both dismantlable if and only if the lattice and its mirror lattice are both planar.

To accomplish this, we use the wealth of existing results on bipartite graphs. We first give a short proof that the bipartite graph family corresponding to dismantlable lattices is the well-studied class of chordal-bipartite graphs (bipartite graphs with no chordless cycle on more than 4 vertices). We then examine the chain dimension of these graphs, to show that when both the graph and its mirror are chordal-bipartite, then both graphs are of chain dimension at most 2, which corresponds to planar lattices.

The paper is organized as follows: Section 2 gives necessary preliminary notations and results on graphs and lattices. Section 3 gives a short proof of the property that a lattice is dismantlable if and only if the corresponding bipartite graph is chordal-bipartite. We give algorithmic considerations on dismantlable lattices in Section 4. Section 5 shows that both the lattice and its mirror lattice are dismantlable if and only if both the lattice and its mirror lattice are planar. We conclude in Section 6.

2 Preliminaries

As our results pertain to both lattices and graphs, we give the necessary notions for both fields.

2.1 Relations, concepts and lattices

Given a finite set $\mathcal{O}$ of *objects* (which we will denote by numbers in our examples) and a finite set $\mathcal{A}$ of *attributes*, (which we will denote by lowercase letters), we will consider a binary *relation* $\mathcal{R}$ as a subset of the Cartesian product $\mathcal{O}\times\mathcal{A}$. The *mirror relation* of $\mathcal{R}$ is the complement relation $\overline{\mathcal{R}} \subseteq \mathcal{O}\times\mathcal{A}$ such that $(x,y)\in\overline{\mathcal{R}}$ *iff* $(x,y)\notin\mathcal{R}$.

$\mathcal{R}(x) = \{y \in \mathcal{A} \,|\, (x,y) \in \mathcal{R}\}$ is the *row* of $x \in \mathcal{O}$ and $\mathcal{R}^{-1}(y) = \{x \in \mathcal{A} \,|\, (x,y) \in \mathcal{R}\}$ is the *column* of y. Rows and columns are both called *lines* of $\mathcal{R}$. A relation is said to be *clarified* when it has no identical lines. A relation is said to be *reduced* when it is clarified and has no line which is the intersection of several other lines.

The triple $(\mathcal{O},\mathcal{A},\mathcal{R})$ is called a *context* [13]; a *concept* of this context is a maximal Cartesian sub-product $X\times Y\subseteq\mathcal{R}$, denoted (X,Y): $\forall x\in X,\ \forall y\in Y,\ (x,y)\in\mathcal{R}$, and $\forall x\in\mathcal{O}-X\ \exists y'\in Y\,|\,(x,y')\notin\mathcal{R}$, and $\forall y\in\mathcal{A}-Y\ \exists x'\in X\,|\,(x',y)\notin\mathcal{R}$. X is called the *extent* of the concept (X,Y), and Y its *intent*. In our examples, we will shorten the notations using for instance $(12, abcde)$ instead of $(\{1,2\},\{a,b,c,d,e\})$.

A *lattice* is a partially ordered set in which every pair $\{e,e'\}$ of elements has both a least upper bound $join(e,e')$ and a greatest lower bound $meet(e,e')$. An element x of a lattice is said to be irreducible if it is either *meet irreducible*: $x{=}meet(e,e') \implies e{=}e'{=}x$, or *join irreducible*: $x{=}join(e,e') \implies e{=}e'{=}x$. A doubly-irreducible element is both meet irreducible and join irreducible. A finite lattice has two *extremal* elements: a lowest element, called the *bottom* element, and a greatest element, called the *top* element.

A lattice is graphically represented by its *Hasse diagram*: transitivity and reflexivity arcs are omitted, and the orientation from bottom to top is implicit. A *maximal chain* of a lattice is a path (all the elements are pairwise comparable) from bottom to top in the Hasse diagram. A *chain lattice* is a lattice which is a chain. A lattice is called *planar* if its Hasse diagram can be represented without crossing edges.

The concepts of a context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ are ordered by inclusion of their intents: $(X, Y) < (X', Y')$ *iff* $X \subset X'$ *iff* $Y' \subset Y$. This defines a finite lattice called a *concept lattice* (or Galois lattice [9]) denoted $\mathcal{L}(\mathcal{R})$. A predecessor of C in $\mathcal{L}(\mathcal{R})$ is any $C' < C$ such that there is no C'' with $C' < C'' < C$; successors are defined dually.

An *object-concept* is a concept C_x which *introduces* some object x: x is in the extent of C_x but is not in the extent of any smaller concept $C' < C_x$. Dually, an *attribute-concept* is a concept C_y which *introduces* some attribute y: y is in the intent of C_y but is not in the intent of any greater concept $C' > C_y$. Thus, the intent of object-concept C_x is $\mathcal{R}(x)$, and the extent of attribute-concept C_y is $\mathcal{R}^{-1}(y)$. Object-concepts and attribute-concepts are also called *introducers*. Objects are introduced from bottom to top and attributes from top to bottom in $\mathcal{L}(\mathcal{R})$. A given concept may introduce several objects and/or attributes. We will call *mixed introducer* a concept which introduces at least one object and at least one attribute.

When a relation is reduced, the irreducible elements of the lattice are exactly the introducers; in a non-reduced but clarified relation, a meet irreducible element introduces exactly one object, and a join irreducible element introduces exactly one attribute.

Our lattices are drawn with the program 'Concept Explorer' [1] using the reduced labeling, where in the Hasse diagrams, each object or attribute labels only one concept: its introducer.

The reader is referred to [13] and [9] for details on lattices and ordered sets.

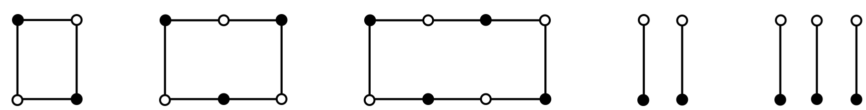
2.2 Graphs

An undirected finite graph is denoted $G = (V, E)$, where V is the vertex set, $|V| = n$, and E is the edge set, $|E| = m$. An edge $\{x, y\} \in E$, linking vertices x and y, is denoted xy; we say that x and y *see* each other or are *adjacent*. A *stable set* is a set of pairwise non-adjacent vertices. The *neighborhood* $N_G(x)$ of a vertex x in graph G is the set of vertices $y \neq x$ such that xy is an edge of E; the subscript G may be omitted. The neighborhood of a set X of vertices is $N(X) = (\bigcup_{x \in X} N(x)) - X$. $G(X)$ denotes the *subgraph induced* by X in G, *i.e.* the subgraph of G with vertex set X and edge set $\{xy \in E \mid x, y \in X\}$.

2.3 Bipartite graphs

A *bipartite graph* $G = (V_1 + V_2, E)$ is a graph whose vertex set can be bipartitioned into two disjoint sets V_1 and V_2, each inducing a stable set. We will call the *mirror* (or bipartite complement) of a bipartite graph $G = (V_1 + V_2, E)$ the bipartite graph $mir(G) = (V_1 + V_2, E')$ such that $\forall x \in V_1$, $\forall y \in V_2$, $xy \in E'$ *iff* $xy \notin E$.

We will say that vertex $x \in V_1$ (resp. $\in V_2$) is *universal* if x sees all the vertices of V_2 (resp. V_1). A *biclique* $(X + Y)$ in a bipartite graph, with $X \subseteq V_1$ and $Y \subseteq V_2$, is defined as having all possible edges: $\forall x \in X, \forall y \in Y,\ xy \in E$. We will refer to two vertices of a bipartite graph as *twin vertices* if they have the same non-empty neighborhood: t and t' are twin vertices if $N(t) = N(t')$ and $N(t) \neq \emptyset$; note that t and t' then both belong to V_1 or both belong to V_2. A C_4 is an induced chordless cycle on 4 vertices, and, more generally, a C_i is an induced chordless cycle on i vertices; an iK_2 is i pairs of adjacent vertices which are pairwise edge-disjoint. These structures are illustrated below.



a C_4 a C_6 a C_8 a $2K_2$ a $3K_2$

An edge xy of a bipartite graph is called *bisimplicial* if $N(x) \cup N(y)$ is a maximal biclique [14], [16].

A bipartite graph is said to be *chordal-bipartite* if it has no chordless induced cycle on strictly more than 4 vertices [14]. A *chain graph* is a bipartite graph with no induced $2K_2$; a chain graph is chordal-bipartite.

Property 1. *[14] A chordal-bipartite graph G has at least one bisimplicial edge e, and removing e from G yields a chordal-bipartite graph.*

Characterization 2. *[14] A bipartite graph is chordal-bipartite iff one can repeatedly remove a bisimplicial edge until no edge is left.*

The reader is referred to [22] and [7] for details on graphs.

2.4 Concepts lattices and bipartite graphs

Any context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ is associated with a bipartite graph $bip(\mathcal{R}) = (\mathcal{O} + \mathcal{A}, E)$, where $xy \in E$ iff $(x, y) \in \mathcal{R}$, and with a concept lattice $\mathcal{L}(\mathcal{R})$. Thus, for $x \in \mathcal{O}$, $N_{bip(\mathcal{R})}(x) = \mathcal{R}(x)$, and for $y \in \mathcal{A}$, $N_{bip(\mathcal{R})}(y) = \mathcal{R}^{-1}(y)$; $\mathcal{O}$ and $\mathcal{A}$ are stable sets of $bip(\mathcal{R})$. $bip(\mathcal{R})$ is a chain graph iff $\mathcal{L}(\mathcal{R})$ is a chain lattice [4]. The bipartite graph associated with the mirror relation $\overline{\mathcal{R}}$ of $\mathcal{R}$, denoted $bip(\overline{\mathcal{R}})$, is the mirror of $bip(\mathcal{R})$.

Property 3. *For any context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$, (X, Y) is a concept of $\mathcal{L}(\mathcal{R})$ iff $X \cup Y$ defines a maximal biclique of $bip(\mathcal{R})$.*

The introducers of a lattice are trivially characterized in their graph counterpart as follows:

Lemma 4. *Let $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ be a context, let (X, Y) be an element of $\mathcal{L}(\mathcal{R})$; then (X, Y) introduces $x \in X$ (resp. $y \in Y$) if and only if $N_{bip(\mathcal{R})}(x) = Y$ (resp. $N_{bip(\mathcal{R})}(y) = X$).*

In the rest of this paper, bipartite graphs denoted $bip(\mathcal{R})$ will implicitly refer to the associated relation $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{A}$ and lattice $\mathcal{L}(\mathcal{R})$.

Example 1. We will use our running example from [5]. Figure 1 shows a relation $\mathcal{R}$ with its associated bipartite graph $bip(\mathcal{R})$, and the associated concept lattice $\mathcal{L}(\mathcal{R})$, as well as the mirror objects associated with $\mathcal{R}$: the complement relation $\overline{\mathcal{R}}$ with its associated graph $bip(\overline{\mathcal{R}})$, and the associated concept lattice $\mathcal{L}(\overline{\mathcal{R}})$.

$\mathcal{R}$	a	b	c	d
1	×	×		×
2		×	×	
3	×			
4			×	

$\mathcal{R}$ and its mirror $\overline{\mathcal{R}}$

$\overline{\mathcal{R}}$	a	b	c	d
1			×	
2	×			×
3		×	×	×
4	×	×		×

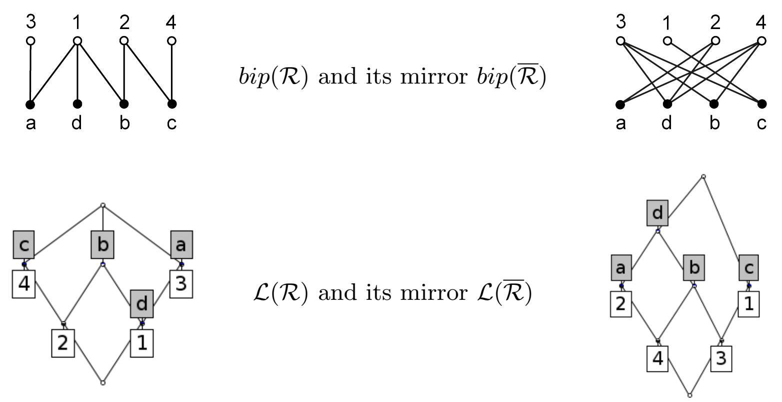


Fig. 1. A relation $\mathcal{R}$, its mirror $\overline{\mathcal{R}}$, the associated graphs and lattices.

3 Dismantlable lattices and chordal-bipartite graphs

Rival [20] studied dismantlable lattices, and proved that when a doubly irreducible is removed from a lattice, a sublattice is obtained.

In order to discuss dismantlability of concept lattices, we will first specify how the relation is to be modified in order to remove a doubly irreducible element from the corresponding lattice.

Theorem 5. *Let (X, Y) be a doubly irreducible element of $\mathcal{L}(\mathcal{R})$, introducing the set of objects $W \subset X$ and the set of attributes $Z \subset Y$; let $\mathcal{R}'$ be the relation obtained by removing from $\mathcal{R}$ all the elements of the Cartesian product $W \times Z$: $\mathcal{R}' = \mathcal{R} - \{(w, z) \in \mathcal{R} \mid w \in W,\, z \in Z\}$.*

Then $\mathcal{L}(\mathcal{R}')$ is a sublattice of $\mathcal{L}(\mathcal{R})$, with the same elements as $\mathcal{L}(\mathcal{R})$, except for the doubly irreducible element (X, Y), which has been removed.

Proof: Let concept (X', Y') be the unique predecessor of doubly irreducible concept (X, Y) and let concept (X'', Y'') be its unique successor.

[20] stated that removing a doubly irreducible element in a lattice results into a sublattice of the initial lattice; in our case, after the removal of (X, Y), (X'', Y'') will become the new successor of (X', Y') and all the other least upper bounds and greatest lower bounds will be preserved.

Thus, for every object $w \in W \subseteq X$ and every attribute $z \in Z \subseteq Y$ introduced by (X, Y), w is in the extents of (X'', Y'') and of all its greater concepts but z is not in their intents; then the removal of (w, z) in $\mathcal{R}$ will not change these greater concepts. The same applies for (X', Y') and its smaller concepts. The other concepts, which are not comparable with (X, Y), also remain unchanged.

After the removal of $W \times Z$, the objects of W will be introduced by (X'', Y'') and the attributes of Z will be introduced by (X', Y') in the new (sub-)lattice $\mathcal{L}(\mathcal{R}')$. $\square$

In order to prove our main theorem in Section 5, we need the result that the class of dismantlable lattices corresponds to the class of chordal-bipartite graphs.

This result was stated by Lifeng Li [17], but to our knowledge there is no available proof, published or otherwise. This result could easily be derived from the characterization of [21] for dismantlable lattices as being crown-free, as a crown in the lattice can be associated with a chordless cycle of length 6 or more in the bipartite graph; it could also be derived from [8], as they show the relationship between strongly chordal graphs and dismantlable lattices, and there is a one-to-one correspondence between strongly chordal graphs and chordal-bipartite graphs [7]. We will prefer a short direct proof using results on bipartite graphs, which will give us some insight on what happens in the graph when a doubly irreducible of the lattice is removed.

We can establish the relationship between bisimplicial edges and mixed introducers:

Property 6. *Let (X, Y) be a doubly irreducible element of $\mathcal{L}(\mathcal{R})$, introducing $x \in X$ and $y \in Y$; then xy is a bisimplicial edge of $bip(\mathcal{R})$.*

Proof: By Lemma 4, in *bip*$(\mathcal{R})$, $N(x) = Y$ and $N(y) = X$. Since (X, Y) is a concept, $N(x) \cup N(y)$ is a maximal biclique, so by definition, xy is a bisimplicial edge. $\square$

Property 7. *Let xy be a bisimplicial edge of $bip(\mathcal{R})$, with $X = N(x)$ and $Y = N(y)$; then (X, Y) is an element of $\mathcal{L}(\mathcal{R})$ introducing both x and y.*

Proof: By definition of a bisimplicial edge xy, $N(x) \cup N(y)$ is a maximal biclique. No vertex outside of $N(x) \cup N(y) \cup \{x, y\}$ can see either x or y, so by Lemma 4, (X, Y) introduces both x and y. $\square$

We can now derive the characterization for dismantlable lattices:

Characterization 8. *A concept lattice is dismantlable iff the associated bipartite graph is chordal-bipartite.*

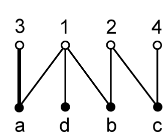
Proof:
Let us consider a dismantlable lattice $\mathcal{L}(\mathcal{R})$. Removing a doubly irreducible element (X, Y) from $\mathcal{L}(\mathcal{R})$ corresponds, by Property 6, to removing a bisimplicial edge from the associated bipartite graph $bip(\mathcal{R})$ if $\mathcal{R}$ is clarified; if it is not, let K be the set of objects which are introduced by (X, Y), and let Z be the set of attributes which are introduced. We can remove all elements of $K \times Z$ by removing bisimplicial edges as follows: choose $x \in K$ and remove all edges in Z which are incident to x, then choose another element of K and remove all edges incident to it in $K \times Z$, an so on.

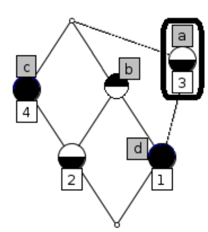
If we go on removing doubly irreducible elements from the lattice, until we obtain a chain lattice, this will thus correspond to a succession of removals of bisimplicial edges from the chordal-bipartite graph until it is a chain graph, which is chordal-bipartite. Since we have eliminated edges from $bip(\mathcal{R})$ and found a chordal-bipartite graph, by Characterization 2, the original bipartite graph was also chordal-bipartite.

Conversely, let $bip(\mathcal{R})$ be a chordal-bipartite graph. If $\mathcal{R}$ is not a reduced relation, let us reduce it, obtaining relation $\mathcal{R}'$; the corresponding bipartite graph $bip(\mathcal{R}')$ is chordal-bipartite, as it was obtained from a chordal-bipartite graph by removing vertices and their incident edges. Thus $bip(\mathcal{R}')$ has a bisimplicial edge xy, which by Property 7 introduces both x and y; since the relation is reduced, the corresponding concept must be a doubly irreducible, which is removed from $\mathcal{L}(\mathcal{R})$. By Characterization 2, the chordal-bipartite graph has then an elimination scheme on bisimplicial edges, so we can repeat this step until the bipartite graph becomes a chain graph, and the corresponding lattice a chain lattice, so the lattice is indeed dismantlable. $\square$

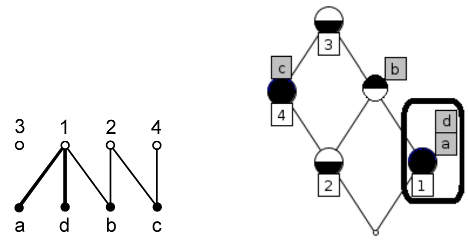
Example 2. Using the relation from Figure 1, the dismantlable scheme is illustrated below. In the lattice, we will successively remove doubly irreducibles labeled: $3a$, $1ad$, $1b$, and $2b$, thereby obtaining a chain lattice.

$3a$ *to be removed:*

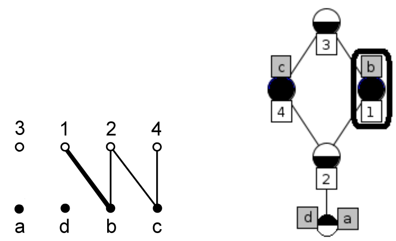




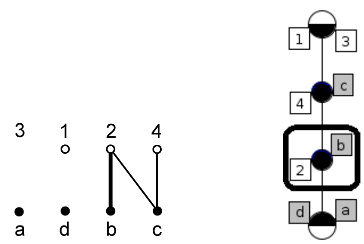
1ad to be removed:



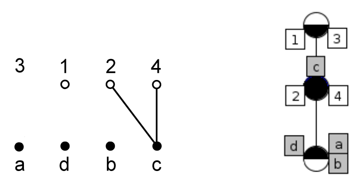
1b to be removed:



2b to be removed:



Finally:



4 Algorithmic aspects

We will now examine how fast we can recognize that a given context corresponds to a dismantlable lattice, using graph results.

Properties 6 and 7 can be extended to characterize the bisimplicial edges of a chordal-bipartite graph as corresponding to irreducible elements or non-irreducible introducers as follows:

Property 9. *Let bip*$(\mathcal{R})$ *be a chordal-bipartite graph with no universal vertex, on vertex set* $V = \mathcal{O} + \mathcal{A}$*; let* xy *be a bisimplicial edge of bip*$(\mathcal{R})$*, with* $x \in \mathcal{O}$ *and* $y \in \mathcal{A}$*; let* $Y = N(x)$*, let* $X = N(y)$*; let* $W = V - (X \cup Y)$*; furthermore, let* $Y' \subset Y$ *be the set of vertices of* Y *which do not see* W *(i.e.* $Y' = \{y \in Y \mid N(y) \cap W = \emptyset\}$*), let* $X' \subset X$ *be the set of vertices of* X *which do not see* W*.*

Then:

1. $\forall x' \in X'$*,* x *and* x' *are twin vertices, and thus define the same line in the table of* $\mathcal{R}$*; likewise,* $\forall y' \in Y'$*,* y *and* y' *are twin vertices.*

2. (X, Y) *is a concept introducing all objects in* X' *and all attributes in* Y'.
3. (X, Y) *is a meet irreducible element iff there is some vertex* y'' *in* W *which sees all the vertices of* $X - X'$, *and likewise* (X, Y) *is a join irreducible element iff there is some vertex* x'' *in* W *which sees all the vertices of* $Y - Y'$.

Proof:

- $X - X' \neq \emptyset$ and $Y - Y' \neq \emptyset$ since there are no universal vertices.
- For $x' \in X', N(x') = Y$, so x and x' are twin vertices (and likewise for y and $y' \in Y'$).
- By Property 7, (X, Y) introduces x and y; since for $x' \in X''$ x and x' are twin vertices, (X, Y) introduces x', and likewise introduces any $y' \in Y''$.
- If there is some x'' which sees all of Y', then x cannot be the intersection of a set of objects, since x'' fails to see y, thus $\mathcal{R}$ is reduced w.r.t. x, so (X, Y), which introduces x, must be meet irreducible. The same reasoning applies for y as join irreducible.
- A line x is the intersection of a set A of other lines iff $N(x) = \bigcap_{z \in A} N(z)$. Let $A = X - X'$; $N(x) = \bigcap_{z \in A} N(z)$ iff there is no vertex y'' in W which sees all the vertices of $X - X'$. Thus (X, Y) is a meet irreducible element iff there is some vertex y'' in W which sees all the vertices of $X - X'$, and likewise (X, Y) is a join irreducible element iff there is some vertex x'' in W which sees all the vertices of $Y - Y'$.

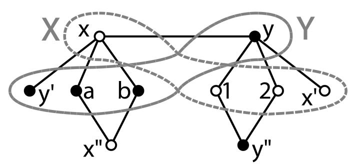


Illustration for the proof of Property 9.

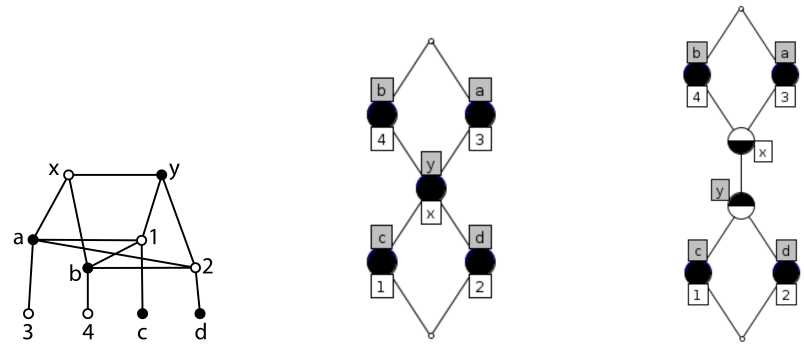
□

As a consequence of Property 9, any bisimplicial edge xy corresponds to a mixed introducer; either x or y can be removed by reducing the relation, or $N(x) \cup N(y)$ corresponds to a doubly irreducible element of $\mathcal{L}(\mathcal{R})$.

As remarked above, removing a doubly irreducible from a lattice will always define a sublattice [20]; however, when one removes the bisimplicial edge which corresponds to a mixed introducer which is *not* irreducible, one does not obtain a sublattice, as illustrated below.

Example 3. Figure 2 shows a chordal-bipartite graph which has a bisimplicial edge xy, as well as the lattices obtained before and after the removal of xy; in the first lattice the concept labeled xy is not irreducible; the second lattice is not a sublattice of the first lattice.

Fortunately, we have tools which enable us to avoid using such a bisimplicial edge. In fact, in order to preserve a sublattice, it is sufficient to eliminate a bisimplicial edge which corresponds to an irreducible element, as summarized below:



Bisimplicial edge xy *Before removing xy* *After removing xy*

Fig. 2. Removing a bisimplicial edge does not necessarily produce a sublattice.

Property 10. *Let xy be a bisimplicial edge, let $Y = N(x)$, let $X = N(y)$, let (X, Y) be the corresponding concept. Then:*

- *If (X, Y) is a doubly irreducible element, the removal of (x, y) from $\mathcal{R}$ removes concept (X, Y).*
- *If (X, Y) is a meet irreducible element but not a doubly irreducible element, introducing x, then the removal of (x, y) from $\mathcal{R}$ will cause y to disappear from the label of concept (X, Y), which thus becomes $(X, Y - \{y\})$, which remains a meet irreducible element introducing x; all the other labels remain unchanged and the lattice is preserved.*
- *If (X, Y) is a join irreducible element but not a doubly irreducible element, introducing y, then the removal of (x, y) from $\mathcal{R}$ will cause x to disappear from the label of concept (X, Y), which thus becomes $(X - \{x\}, Y)$, which remains a join irreducible element introducing y; all the other labels remain unchanged and the lattice is preserved.*

Results on chordal-bipartite graphs enable us to repeatedly find a bisimplicial edge which is a join irreducible element very efficiently, using the following well-known characterization of chordal-bipartite graphs:

Characterization 11. *A bipartite graph is chordal-bipartite iff its matrix can be arranged so that it contains no Γ (a Γ is a 2×2 submatrix with the unique 0 entry at the lower right-hand corner).*

Example 4. The matrices of relations $bip(\mathcal{R})$ and $bip(\overline{\mathcal{R}})$ from Figure 1 may be reordered into Γ-free matrices:

$\mathcal{R}$	a	d	b	c
3	×			
1	×	×	×	
2			×	×
4				×

$\overline{\mathcal{R}}$	c	b	d	a
1	×			
3	×	×		
4		×	×	×
2				×

The first non-zero entry of such a Γ-free matrix will yield a desirable bisimplicial edge:

Lemma 12. *[22] Let M be a Γ-free matrix of a chordal-bipartite graph $bip(\mathcal{R})$, let x be the object which is the first row of M, let Y be the set of neighbors of x in $bip(\mathcal{R})$; then the neighborhoods of the attributes in Y can be totally ordered by inclusion, and this ordering corresponds to the ordering on the columns of M.*

Note that dually, the attribute y which is the first column of a Γ-free matrix will have a similar ordering on the neighbors of y.

Note also that this neighborhood inclusion is reminiscent of the 'simple vertices' of strongly chordal graphs used for the same purpose in [8] to dismantle co-atomistic dismantlable lattices.

As a consequence of Lemma 12, Property 9 and Property 10, the first non-zero entry on the first row of a Γ-free matrix will define a bisimplicial edge which corresponds to a join irreducible element.

Thus, because removing the first non-zero entry of a Γ-free matrix preserves the Γ-free property, one can derive from a Γ-free matrix an elimination scheme on join irreducible elements of the corresponding dismantlable lattice. At each step eliminating a bisimplicial edge, either the structure of the lattice remains unchanged, or a doubly irreducible element is removed from the lattice.

Of course, dually choosing the entries of the first column and traversing them from top to bottom before going on to the second column and so forth will yield an ordering on bisimplicial edges which correspond to meet irreducible elements.

Example 5. In Example 2, the ordering illustrated on the elimination of doubly irreducibles from $\mathcal{L}(\mathcal{R})$ is the one suggested by the corresponding Γ-free matrix from Example 4. Notice how edges $1a$ and $1d$ are removed simultaneously with a doubly irreducible introducing $1, a$ and d, because the previous removal of edge $3a$ has made a and d twin vertices.

From the results discussed above, we could deduce the already known property that a dismantlable lattice $\mathcal{L}(\mathcal{R})$ has at most $|\mathcal{R}|$ elements.

Chordal-bipartite graphs with n vertices and m edges can be recognized in $O(\min(n^2, m\,logn))$ time [18, 19, 23] by computing a matrix with a 'doubly lexical ordering'; the graph is chordal-bipartite if and only if this matrix has no Γ. Thus an elimination scheme on 'good' bisimplicial edges of a chordal-bipartite graph $bip(\mathcal{R})$ can be found in $O(\min(n^2, m\,logn))$ time, where $n = |\mathcal{O}| + |\mathcal{A}|$, and $m = |\mathcal{R}|$.

Given a Γ-free matrix, the maximal bicliques can be computed in time $O(n+m)$ [16] using the ordering from left to right and from top to bottom suggested by the Γ-free matrix; a cheap pre-processing step enables the user to decide which new sets of twin vertices appear during the elimination process. However, the maximal bicliques are not, in general, computed in an order corresponding to a doubly irreducible elimination scheme of the lattice. In a reduced relation, every bisimplicial edge corresponds to a doubly irreducible element of the lattice; if we reduce the relation after each elimination step of a bisimplicial edge, we will find an elimination scheme on doubly irreducible elements. Doing this in a straightforward fashion would however be more costly than computing all the maximal bicliques, constructing the lattice, and finding doubly irreducible elements in the Hasse diagram.

5 Dismantlable lattices whose mirror is dismantlable

We shall now investigate lattices which are dismantlable and whose mirror lattice is also dismantlable. Rival [20] stated the following result on dismantlable lattices:

Lemma 13. *[3] Every planar lattice is dismantlable.*

We will show a stronger relationship between planar lattices and dismantlable lattices:

Theorem 14. *Let $\mathcal{R}$ be a binary relation associated with the concept lattice $\mathcal{L}(\mathcal{R})$ and the bipartite graph bip$(\mathcal{R})$; let $\overline{\mathcal{R}}$ be the mirror relation associated with the concept lattice $\mathcal{L}(\overline{\mathcal{R}})$ and the bipartite graph bip$(\overline{\mathcal{R}})$.*
Then the following are equivalent:

(1) $\mathcal{L}(\mathcal{R})$ is a dismantlable lattice and its mirror lattice $\mathcal{L}(\overline{\mathcal{R}})$ is also a dismantlable lattice.
(2) $\mathcal{L}(\mathcal{R})$ is a planar lattice and its mirror lattice $\mathcal{L}(\overline{\mathcal{R}})$ is also a planar lattice.
(3) bip$(\mathcal{R})$ is chordal-bipartite and its mirror bipartite graph bip$(\overline{\mathcal{R}})$ is also chordal-bipartite.

(1) is equivalent to (3) by Characterization 8. We will show that (3) is equivalent to (2).

We will need some extra definitions and properties on bipartite graphs:

Definition 15.
*The **chain dimension** of a bipartite graph bip$(\mathcal{R})$ is the minimum number of chain graphs which give bip$(\mathcal{R})$ as their intersection [22].*
*The **chain cover number** of a bipartite graph bip$(\mathcal{R})$ is the minimum number of chain graphs needed to cover the edge set of bip$(\mathcal{R})$.*

Clearly, the chain dimension of a bipartite graph *bip*$(\mathcal{R})$ is the chain cover number of its mirror *bip*$(\overline{\mathcal{R}})$ and vice-versa.

Theorem 16. *[3, 6, 11, 22, 12] Let $\mathcal{R}$ be a binary relation; then $\mathcal{L}(\mathcal{R})$ is a planar lattice if and only if the chain dimension of the corresponding bipartite graph bip$(\mathcal{R})$ is at most 2.*

Thus $\mathcal{L}(\mathcal{R})$ is a planar lattice iff *bip*$(\overline{\mathcal{R}})$ can be covered by at most 2 disjoint chain graphs. Abueida, Busch and Sritharan [2] studied the chain cover number of a bipartite graph. In particular, they showed the following result:

Property 17. *[2] If bip$(\mathcal{R})$ is a chordal-bipartite graph, then the chain cover number of bip$(\mathcal{R})$ is equal to the size of a largest induced matching.*

An *induced matching* is a set of edges such that no two edges are joined by an edge in the graph; as a result, an induced matching of size i corresponds to an induced iK_2.

Now when a chordal-bipartite graph has a mirror which is also chordal-bipartite, it can have no induced $3K_2$:

Lemma 18. *$bip(\mathcal{R})$ is chordal-bipartite with no induced $3K_2$ iff its mirror bipartite graph $bip(\overline{\mathcal{R}})$ is also chordal-bipartite with no induced $3K_2$.*

Proof: Let $bip(\mathcal{R})$ be a chordal-bipartite graph with no induced $3K_2$; the mirror of a $3K_2$ is an induced C_6 (a chordless cycle on 6 vertices), but $bip(\mathcal{R})$ by definition of a chordal-bipartite graph has no C_6, so $bip(\overline{\mathcal{R}})$ has no $3K_2$; suppose $bip(\overline{\mathcal{R}})$ fails to be chordal-bipartite; any chordless induced cycle of length 10 or more contains a $3K_2$, so $bip(\overline{\mathcal{R}})$ must have a C_6 or a C_8 (bipartite graphs have only even cycles); if $bip(\overline{\mathcal{R}})$ has a C_6, $bip(\mathcal{R})$ has a $3K_2$, which is impossible by our hypothesis; the mirror of a C_8 is a C_8, so if $bip(\overline{\mathcal{R}})$ has a C_8, $bip(\mathcal{R})$ has a C_8, which is impossible since $bip(\mathcal{R})$ is chordal-bipartite. □

Corollary 19. *If $bip(\mathcal{R})$ is a chordal-bipartite graph with no $3K_2$ then $\mathcal{L}(\overline{\mathcal{R}})$ is a planar lattice.*

Proof: Let $bip(\mathcal{R})$ be a chordal-bipartite graph with no $3K_2$; by Property 17, then the chain cover number of $bip(\mathcal{R})$ is at most 2; by Theorem 16, $\mathcal{L}(\overline{\mathcal{R}})$ is a planar lattice. □

Combining this with the results presented above, we obtain the following:

Theorem 20. *The following are equivalent:*

(1) $\mathcal{L}(\mathcal{R})$ is a planar lattice and its mirror lattice $\mathcal{L}(\overline{\mathcal{R}})$ is also a planar lattice
(2) $bip(\mathcal{R})$ is chordal-bipartite and its mirror bipartite graph $bip(\overline{\mathcal{R}})$ is also chordal-bipartite.

Proof: Let $bip(\mathcal{R})$ be a chordal-bipartite graph whose mirror is also a chordal-bipartite graph. $bip(\mathcal{R})$ has no $3K_2$, so by Lemma 18, both $bip(\mathcal{R})$ and $bip(\overline{\mathcal{R}})$ are chordal-bipartite with no induced $3K_2$; by Corollary 19, both $\mathcal{L}(\mathcal{R})$ and $\mathcal{L}(\overline{\mathcal{R}})$ are planar lattices.

Conversely, if both $\mathcal{L}(\mathcal{R})$ and $\mathcal{L}(\overline{\mathcal{R}})$ are planar lattices, then by Lemma 13, $\mathcal{L}(\mathcal{R})$ and $\mathcal{L}(\overline{\mathcal{R}})$ are dismantlable, and by Characterization 8 $bip(\mathcal{R})$ and $bip(\overline{\mathcal{R}})$ are both chordal-bipartite. □

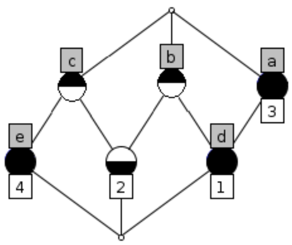
Example 6. In Figure 1 from Example 1, both $bip(\mathcal{R})$ and $bip(\overline{\mathcal{R}})$ are chordal-bipartite, and both $\mathcal{L}(\mathcal{R})$ and $\mathcal{L}(\overline{\mathcal{R}})$ are planar lattices.

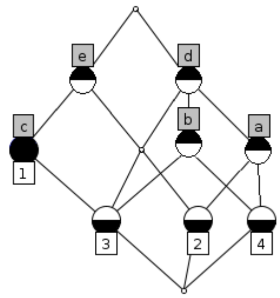
If, however, we add an element $(4, e)$ to the relation, obtaining the new relation $\mathcal{R}'$, $bip(\mathcal{R}')$ remains chordal-bipartite, but it contains a $3K_2$: $\{3a, 2b, 4e\}$; its lattice $\mathcal{L}(\mathcal{R}')$ is dismantlable and planar, but the mirror lattice $\mathcal{L}(\overline{\mathcal{R}}')$ is neither dismantlable nor planar.

$\mathcal{R}$	a	b	c	d	e
1	×	×		×	
2		×	×		
3	×				
4			×		×

$\mathcal{R}'$ and its mirror $\overline{\mathcal{R}'}$

$\overline{\mathcal{R}}$	a	b	c	d	e
1			×		×
2	×			×	×
3		×	×	×	×
4	×	×		×	





$\mathcal{L}(\mathcal{R}')$ and its mirror $\mathcal{L}(\overline{\mathcal{R}}')$

Fig. 3. A relation $\mathcal{R}'$ which defines a chordal-bipartite graph which contains a $3K_2$, its mirror $\overline{\mathcal{R}'}$, the associated lattices $\mathcal{L}(\mathcal{R}')$ and its mirror $\mathcal{L}(\overline{\mathcal{R}}')$. $\mathcal{L}(\overline{\mathcal{R}}')$ fails to be planar and dismantlable.

Definition 21. *We will define as* auto-dismantlable *a lattice which is dismantlable and whose mirror lattice is also dismantlable, and we will likewise define the notions of* auto-planar *lattice and* auto-chordal-bipartite *graph.*

With the results from Section 4, auto-dismantlable and auto-planar lattices can be recognized in $O(n^2)$ time, where $n = |\mathcal{O}| + |\mathcal{A}|$.

6 Conclusion and perspectives

We have characterized the class of relations which correspond to dismantlable concept lattices as defining chordal-bipartite graphs. We have uncovered a strong connection between dismantlability and planarity, by showing that a lattice is auto-dismantlable if and only if it is auto-planar.

Using relation $\mathcal{R}$, we can decide in $O((|\mathcal{O}| + |\mathcal{A}|)^2)$ time whether $\mathcal{L}(\mathcal{R})$ is a dismantlable lattice; we leave open the question of defining an elimination scheme on doubly irreducible elements of $\mathcal{L}(\mathcal{R})$ in $O((|\mathcal{O}| + |\mathcal{A}|)^2)$ time.

Both the relations in Example 4 have the 'consecutive ones' property (the binary matrix can be ordered so that on each row, the 'ones' are consecutive); the corresponding bipartite graph is chordal-bipartite and is called a convex graph; [12] showed that relations with the consecutive ones property are planar; however, not all chordal-bipartite graphs with no $3K_2$ are convex graphs, and some convex graphs may have a $3K_2$, so convex graphs are not necessarily auto-chordal-bipartite, and the corresponding lattice is not necessarily auto-planar.

Chordal-bipartite graphs are characterized as bipartite graphs from which one can repeatedly remove a vertex which is not the center of a P_5 [15] (a

P_5 is an induced chordless path on 5 vertices); the first row and column of a Γ-free matrix define such vertices [22]. The removal from the relation of the corresponding object or attribute could be interesting to examine.

We have yet to characterize what happens exactly in the mirror lattice $\mathcal{L}(\overline{\mathcal{R}})$ of an auto-dismantlable lattice $\mathcal{L}(\mathcal{R})$ when a doubly irreducible is removed from $\mathcal{L}(\mathcal{R})$.

Finally, the recognition of chordal-bipartite graphs in linear $O(|\mathcal{R}|)$ time is a popular open graph problem [22]. We hope that in the light of dismantlability this problem can be solved.

References

1. http://conexp.sourceforge.net/download.html, release 1.3. © S.A. Yevtushenko & al. 2000-2006. [Yevtushenko S.A.: *System of data analysis "Concept Explorer"* (In Russian). Proc. 7th nat. conf. on Artif. Intell. KII-2000 (Ru), pp127-134, 2000.]
2. Abueida A.A., Busch A.H., Sritharan R.: *A Min-Max Property of Chordal Bipartite Graphs with Applications.* Graphs and Combinatorics, 26:3, pp 301-313 (2010).
3. Baker K.A., Fishburn P.C., Roberts F.S.: *Partial orders of dimension 2, Interval Orders, and Interval Graphs.* Networks, 2:1, pp 11-28 (1972).
4. Berry A., Sigayret A.: *Representing a concept lattice by a graph.* Discrete Applied Mathematics, 144:1-2, pp 27-42 (2004).
5. Berry A., Sigayret A.: *A peep through the looking glass: articulation points in lattices.* Proc.ICFCA'12. LNAI 7278, pp 45-60 (2012).
6. Birkhoff G.: *Lattice Theory.* American Mathematical Society, 3rd Edition (1967).
7. Brandstädt A., Le V.B., Spinrad J.P.: *Graph Classes: A Survey.* SIAM Monographs on Discrete Math. Appl., 3 (1999).
8. Brucker F., Gély A.: *Crown-free Lattices and Their Related Graphs.* Order, 28(3), pp 443-454 (2011).
9. Caspard N., Leclerc B., Monjardet B.: *Ensembles ordonnés finis : concepts, résultats et usages.* Mathémathiques et Applications, 60, Springer (2007).
10. Deiters K., Erné M.: *Negations and contrapositions of complete lattices.* Discrete Mathematics, 181:1-3, pp 91-111 (1998).
11. Dushnik B., Miller E.W.: *Partially ordered sets.* Amer. J. Mathematics, 63, pp 600-610 (1941).
12. Eschen E.M., Pinet N., Sigayret A.: *Consecutive-ones: handling lattice planarity efficiently.* Proc. CLA'07 (2007).
13. Ganter B., Wille R.: *Formal Concept Analysis.* Springer (1999).
14. Golumbic M.M., Goss M.C.: *Perfect Elimination and Chordal Bipartite Graphs.* Journal of Graph Theory, 2:2, pp 155-163 (1978).
15. Hammer P.L., Maffray F., Preissmann M.: *A characterization of chordal bipartite graphs.* RUTCOR Research Report, Rutgers University (New Brunswick, NJ), pp 16-89 (1989).
16. Kloks T., Kratsch D.: *Computing a Perfect Edge Without Vertex Elimination Ordering of a Chordal Bipartite Graph.* Information Processing Letters, 55:1, pp 11-16 (1995).
17. Lifeng Li: *Private communication.*
18. Lubiw, A.: *Doubly lexical orderings of matrices.* SIAM J. Comput., 16:5, pp 854-879, (1987).

19. Paige R., Tarjan R.E.: *Three partition refinement algorithms.* SIAM J. Comput., 16:6, pp 973-989 (1987).
20. Rival I.: *Lattices with doubly irreducible elements.* Canadian Mathematical Bulletin, 17:1, pp 91-95 (1974).
21. Kelly D., Rival I.: *Crowns, fences, and dismantlable lattices.* Canadian J. Mathematics, 26, pp 1257-1271 (1974).
22. Spinrad J.P.: *Efficient Graph Representations.* Fields Institute Monographs, 19, AMS (2003).
23. Spinrad J.P.: *Doubly lexical ordering of dense 0-1 matrices.* Information Processing Letters, 45:5 pp 229-235 (1993).
24. Spinrad J.P., Sritharan R.: *Algorithms for weakly triangulated graphs.* Discrete Applied Mathematics, 19, pp 181-191 (1995).