A peep through the looking glass: articulation points in lattices

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Abstract. We define as an 'articulation point' in a lattice an element which is comparable to all the other elements, but is not extremum.
We investigate a property which holds for both the lattice of a binary relation and for the lattice of the complement relation (which we call the mirror relation): one has an articulation point if and only if the other has one also.
We give efficient algorithms to generate all the articulation points. We discuss artificially creating such an articulation point by adding or removing crosses of the relation, and also creating a chain lattice.
We establish the strong relationships with bipartite and co-bipartite graphs; in particular, we derive efficient algorithms to compute a minimal triangulation and a maximal sub-triangulation of a co-bipartite graph, as well as to find the clique minimal separators and the corresponding decomposition.

Keywords: articulation point; chain lattice; complement relation; co-bipartite graph; minimal triangulation; clique separator decomposition

1 Introduction

In previous papers we showed and exploited the strong relationship between the lattice built on the maximal bicliques of a bipartite graph and the minimal separators of the co-bipartite graph, which is the complement of the bipartite graph [4].

A question which we have often been asked is: "But what can you say about the lattice of the bipartite complement ?"

In this paper, we begin our investigation of this question with a simple property which is common to both lattices: a lattice has an articulation point if and only if the lattice of the bipartite complement has an articulation point. What we call an 'articulation point' in a lattice is an element which is comparable to all

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the other elements, but is not an extremum. The removal of such an 'articulation point' disconnects the lattice diagram.

In order to avoid sentences such as "The co-bipartite graph which is the complement of the bipartite complement ...", we have chosen to refer to the complement relation as the 'mirror relation'. We will investigate properties of the relations, lattices and graphs seen with this 'looking glass'.

We characterize in terms of binary relation the cases where the lattice has an articulation point. We give an efficient algorithm to find all the articulation points of a lattice. We also examine the case where there is no articulation point: we can create one by either adding or removing an inclusion-minimal set of elements of the relation.

We then go on to discuss the case where all the elements of the lattice are articulation points. Such a lattice is a chain, as well as its mirror lattice. We give a linear-time algorithm to recognize a chain lattice, and then use it to embed a lattice into a chain by adding or removing an inclusion-minimal set of crosses of the relation.

In all cases, we explore the relationship with the bipartite and co-bipartite graphs involved. Our approach uses lattice theory to propose an alternate process for several graph algorithms. We also derive a new and more efficient algorithm to compute a minimal triangulation and a maximal subtriangulation of a co-bipartite graph.

The paper is organized as follows: Section 2 gives preliminary notations and results. Sections 3 deals with lattices endowed with an articulation point. Section 4 discusses the algorithmic issues of finding these articulation points efficiently. Section 5 investigates lattices which are chains. Section 5.3 addresses the issues related to artificially creating a lattice which is a chain; in particular we improve the triangulation of a co-bipartite graph. We conclude in Section 6.

2 Preliminaries

As our results are pertaining to both lattices and graphs, we give the necessary notions for both fields.

2.1 Relations, concepts and lattices

Given a finite set $\mathcal{O}$ of *objects* (which we will denote by numbers in our examples) and a finite set $\mathcal{A}$ of *attributes*, (which we will denote by lowercase letters), we will consider a binary **relation** $\mathcal{R}$ as a subset of the Cartesian product $\mathcal{O} \times \mathcal{A}$. We will refer to elements of $\mathcal{R}$ as **crosses**. For $x \in \mathcal{A}$, we will denote $\mathcal{R}(x) = \{y \in \mathcal{A} \,|\, (x,y) \in \mathcal{R}\}$, and for $y \in \mathcal{O}, \mathcal{R}(y) = \{x \in \mathcal{O} \,|\, (x,y) \in \mathcal{R}\}$. For $X \subseteq \mathcal{O}$ and $Y \subseteq \mathcal{A}$, subrelation $\mathcal{R}(X,Y)$ denotes the restriction of $\mathcal{R}$ to X and Y: $(x,y) \in \mathcal{R}(X,Y)$ *iff* $(x,y) \in \mathcal{R}$ and $x \in X$ and $y \in Y$. The **mirror relation** of $\mathcal{R}$ is the relation $\overline{\mathcal{R}} \subseteq \mathcal{O} \times \mathcal{A}$ such that $(x,y) \in \overline{\mathcal{R}}$ *iff* $(x,y) \notin \mathcal{R}$.

The triple $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ is called a **context** [12]; a **concept** of this context is a maximal sub-product $X \times Y \subset \mathcal{R}$, denoted (X,Y): $\forall x \in X,\ \forall y \in Y,\ (x,y) \in \mathcal{R}$, and $\forall x \in \mathcal{O} - X\ \exists y' \in Y \,|\, (x, y') \notin R$, and $\forall y \in \mathcal{A} - Y\ \exists x' \in X \,|\, (x', y) \notin R$. X is called the **extent** of concept (X, Y), and Y its **intent**. In our examples, we will shorten the notations using for instance $(12, abcde)$ instead of $(\{1, 2\}, \{a, b, c, d, e\})$.

A **lattice** is a partially ordered set in which every pair $\{e, e'\}$ of elements has both a lowest upper bound and a greatest lower bound. A finite lattice has two **extremal** elements: a lowest element, called the **bottom** element, and a greatest element, called the **top** element. A lattice is graphically represented by its **Hasse diagram**: transitivity and reflexivity arcs are omitted, and the orientation from bottom to top is implicit. In the Hasse diagrams, only the objects or attributes which appear for the first time are represented, as detailed in the example given in Subsection 2.6. Our lattices are drawn with the program 'Concept Explorer' [1]. A **maximal chain** of a lattice is a path (all the elements are comparable) from bottom to top in the Hasse diagram.

The concepts of a context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ are ordered by inclusion on their intents: $(X, Y) < (X', Y')$ *iff* $X \subset X'$ *iff* $Y' \subset Y$. This defines a finite lattice called a **concept lattice** (or Galois lattice [11]) denoted $\mathcal{L}(\mathcal{R})$. Two concepts (X, Y) and (X', Y') are comparable if $X \subset X'$ or $X' \subset X$. A concept (X', Y') is a descendant of (X, Y) if $X \subset X'$. Concept (X', Y') is said to cover concept (X, Y) if $X \subset X'$ and there is no concept (X'', Y'') such that $X \subset X'' \subset X'$.

For any concept (X, Y), the descendants of (X, Y) form a sub-lattice, which is isomorphic to the lattice formed on Bordat's subrelation [7] $\mathcal{R}(\mathcal{O} - X, Y)$: any concept (W, Z) of this relation corresponds to concept $(W + X, Z)$ of the original relation.

The reader is referred to [12] and [11] for details on lattices and ordered sets.

2.2 Graphs

An undirected finite graph is denoted $G = (V, E)$, where V is the vertex set, $|V| = n$, and $E \subset V^2$ is the edge set, $|E| = m$. The **neighborhood** $N_G(x)$ of vertex x in graph G is the set of vertices $y \neq x$ such that xy is an edge of E (we then say that x and y **see** each other). The neighborhood $N_G(X)$ of a set X of vertices is $N(X) = (\bigcup_{x \in X} N(x)) - X$. $G(X)$ denotes the **subgraph induced** by X in G, *i.e.* the subgraph of G with vertex set X and edges set $\{xy \in E \,|\, x, y \in X\}$.

A **clique** is a set X of vertices with all possible edges (i.e. $\forall x, y \in X, x \neq y, xy \in E$). A **maximal clique module** is a clique X such that $\forall x, y \in X, N(x) \cup \{x\} = N(y) \cup \{y\}$, and which is maximal for this property. A **stable set** (or independent set) is a set X of vertices with no edge (i.e. $\forall x, y \in X, xy \notin E$). A **path** in a graph is a sequence $(x_0, ..., x_k)$ of vertex such that, for any $i \in [0, k[$, $x_i x_{i+1}$ is an edge of the graph. A **cycle** of length k is a path $(x_0, ..., x_k)$ with $x_0 = x_k$ and $k > 2$. A **chord** in a cycle is an edge between two non-consecutive vertices of the cycle. A C_4 is an induced chordless cycle on 4 vertices, a $2K_2$ is the complement of a C_4. A **connected component** of a graph is an inclusion-maximal set of vertices in which there is a path between any pair of distinct vertices. A graph is said to be connected if it has only one connected component, and disconnected otherwise.

The **complement** of graph $G = (V, E)$ is graph $\overline{G} = (V, \overline{E})$ with $\overline{E} = \{xy \,|\, x \neq y \text{ and } xy \notin E\}$.

Minimal separators

A **separator** S of a connected graph $G = (V, E)$ is a subset of vertices the removal of which disconnects the graph; a separator S is called a **minimal separator** if there are at least two connected components X and Y of $G(V-S)$ such that $N(X) = N(Y) = S$.

A separator S is called a **clique separator** if it is a separator and a clique; we will say that we **saturate** a non-clique separator S if we add all missing edges necessary to make S a clique. Clique minimal separator **decomposition** is a graph decomposition which repeatedly uses a clique minimal separator S to replace the current graph $G = (V, E)$ with subgraphs $G(C_i \cup N(C_i))$, where C_i is a connected component of $G(V - S)$; the final set of subgraphs obtained are called '**atoms**' (see [3] for full details on this decomposition). A minimal separator S is said to **cross** another minimal separator S' if S' has at least on vertex in each connected component of $G(V - S)$ [17]. Saturating a minimal separator S causes all the minimal separators which cross S to disappear. Thus a minimal separator is a clique if and only if it crosses no other minimal separator [17], [5].

Chordal graphs and triangulation

A graph is said to be **chordal** (or **triangulated**) if it contains no chordless induced cycle of length strictly greater than three. **Minimal triangulation** is the process of embedding a graph $G = (V, E)$ into a chordal graph $H = (V, E+F)$ by the *addition* of an inclusion-minimal set F of edges: H is chordal but fails to remain chordal if any proper subset of edges $F' \subset F$ is removed. A graph is chordal if and only if all its minimal separators are cliques. Repeatedly saturating a minimal separator is a process which yields a minimal triangulation [5].

A **maximal subtriangulation** $H' = (V, E-F')$ is a chordal graph obtained from graph $G = (V, E)$ by *removing* an inclusion-minimal set of edges.

2.3 Bipartite graphs

A **bipartite graph** $G = (V_1 + V_2, E)$ is a graph whose vertex set can be bipartitioned into two disjoint sets V_1 and V_2, each inducing a stable set. A **biclique** $(X + Y)$ in a bipartite graph, with $X \subseteq V_1$ and $Y \subseteq V_2$, is defined as having all possible edges: $\forall x \in X, \forall y \in Y,\ xy \in E$. We will say that vertex $x \in X$ (resp. $y \in Y$) is **universal** if x sees all the vertices of Y (resp. X).

We will call **mirror** (or bipartite complement) of bipartite graph $G = (V_1 + V_2, E)$ the bipartite graph $mir(G) = (V_1 + V_2, F)$ such that $\forall x \in V_1, y \in V_2$, $xy \in F$ *iff* $xy \notin E$.

Any context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ is associated with bipartite graph $bip(\mathcal{R}) = (\mathcal{O}+\mathcal{A}, E)$, where $xy \in E$ if $(x, y) \in \mathcal{R}$. There is a one-to-one correspondence between the maximal bicliques of $G = bip(\mathcal{R})$ and the concepts of $\mathcal{L}(\mathcal{R})$.

2.4 Co-bipartite graphs

A **co-bipartite** graph is a graph which is the complement of a bipartite graph. The vertex set of a co-bipartite graph can thus be partitioned into two disjoint sets V_1 and V_2, each inducing a clique. Any minimal separator S of a co-bipartite graph defines exactly two connected components, X and Y, with $X \subset V_1$ and $Y \subset V_2$ and $S = N(X) = N(Y)$ [4].

We will call **mirror** of co-bipartite graph $G = (V_1 + V_2, E)$ the co-bipartite graph $mir(G) = (V_1 + V_2, F)$ with the same cliques sets X and Y, and where for $x \in V_1$ and $y \in V_2$, $xy \in F$ *iff* $xy \notin E$.

The reader is referred to [18] and [9] for details on graphs.

2.5 Lattices and co-bipartite graphs

Any context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ is associated with a concept lattice $\mathcal{L}(\mathcal{R})$, a bipartite graph $bip(\mathcal{R})$ built on stable sets $\mathcal{O}$ and $\mathcal{A}$, and a co-bipartite graph $cobip(\mathcal{R})$ built on cliques $\mathcal{O}$ and $\mathcal{A}$, where xy is an external edge of $cobip(\mathcal{R})$ *iff* $xy \notin \mathcal{R}$.

Theorem 1. *[4] Let $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ be a context, let $cobip(\mathcal{R})$ be the corresponding co-bipartite graph. Then (X, Y) is a concept of $\mathcal{R}$ if and only if $S = V - (X \cup Y)$ is a minimal separator of $cobip(\mathcal{R})$, minimally separating $X \subset \mathcal{O}$ from $Y \subset \mathcal{A}$.*

Characterization 1 *[4] Given a context $(\mathcal{O}$,$\mathcal{A}$,$\mathcal{R})$, concepts $(X$,$Y)$ and $(X'$,$Y')$ are comparable elements of $\mathcal{L}(\mathcal{R})$ if and only if their respective associated minimal separators $S = (\mathcal{O} - X) \cup (\mathcal{A} - Y)$ and $S' = (\mathcal{O} - X') \cup (\mathcal{A} - Y')$ are non-crossing minimal separators of $cobip(\mathcal{R})$.*

2.6 Example

Figure 1 shows a relation $\mathcal{R}$ with its associated bipartite graph $bip(\mathcal{R})$, the corresponding co-bipartite graph $cobip(\mathcal{R})$, and the associated concept lattice $\mathcal{L}(\mathcal{R})$, as well as the mirror objects associated with $\mathcal{R}$: the complement relation $\overline{\mathcal{R}}$ with its associated graph $bip(\overline{\mathcal{R}})$, the corresponding co-bipartite graph $cobip(\overline{\mathcal{R}})$, and the associated concept lattice $\mathcal{L}(\overline{\mathcal{R}})$.

3 Lattices with an articulation point

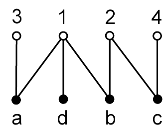
We will first characterize the relations whose lattices are endowed with an articulation point, and then examine how this is translated in the mirror relation.

Definition 1. *Let $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ be a context. A concept (X, Y) which is not the top or bottom element is called an* ***articulation point*** *of $\mathcal{L}(\mathcal{R})$ if it is comparable with all the other elements of $\mathcal{L}(\mathcal{R})$.*

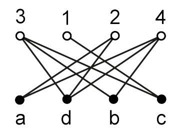
$\mathcal{R}$	a	b	c	d
1	×	×		×
2		×	×	
3	×			
4			×	

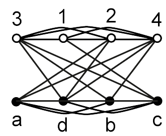
$\mathcal{R}$ and its mirror relation $\overline{\mathcal{R}}$

$\overline{\mathcal{R}}$	a	b	c	d
1			×	
2	×			×
3		×	×	×
4	×	×		×

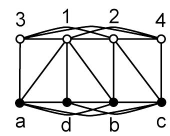


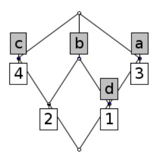
bip($\mathcal{R}$) and its mirror *bip*($\overline{\mathcal{R}}$)





cobip($\mathcal{R}$) and its mirror *cobip*($\overline{\mathcal{R}}$)





$\mathcal{L}(\mathcal{R})$ and its mirror lattice $\mathcal{L}(\overline{\mathcal{R}})$

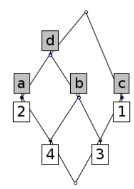


Fig. 1. A relation $\mathcal{R}$, its complement $\overline{\mathcal{R}}$, the associated graphs and lattices.

3.1 Cases where the lattice has an articulation point

Characterization 2 *Let (X, Y) be a concept of $\mathcal{L}(\mathcal{R})$. (X, Y) is an articulation point of $\mathcal{L}(\mathcal{R})$ if and only if in bip($\mathcal{R}$), $(\mathcal{O}-X)\cup(\mathcal{A}-Y)$ is a stable set containing at least one vertex of $\mathcal{O}$ and at least one vertex of $\mathcal{A}$.*

Proof: Let $G = bip(\mathcal{R})$. (X, Y) is an articulation point: by definition, (X, Y) is comparable to all the other concepts, and by Characterization 1, $S = (\mathcal{O} - X) \cup (\mathcal{A} - Y)$ is a clique in co-bipartite graph *cobip*($\mathcal{R}$), and therefore a stable set in *bip*($\mathcal{R}$). There must be at least two concepts (W, Z) with $W \subset X$ and (W', Z') with $Z' \subset Y$, else (X, Y) is extremum. If there is a concept (X, Y) such that $G(V-(X+Y))$ is a stable set containing at least one vertex of $\mathcal{O}$ and at least one vertex of $\mathcal{A}$, then there is at least one element of $\mathcal{L}(G)$ above (X, Y) and at least on element below, so (X, Y) is not extremum; suppose there is a concept (X', Y') which is not comparable with (X, Y): let x be a vertex of $X - X'$, y a vertex of $Y - Y'$; xy is an edge of $G(V-(X+Y))$, which then fails to be a stable set. □

We are now ready to present our main theorem:

Theorem 2. *A concept lattice $\mathcal{L}(\mathcal{R})$ has an articulation point if and only if its mirror concept lattice $\mathcal{L}(\overline{\mathcal{R}})$ has an articulation point.*

Proof: Let $G = (V_1 + V_2, E)$ be a bipartite graph, let $X + Y$ be a maximal biclique of G such that $(V_1 - X) \cup (V_2 - Y)$ induces a stable set. The mirror of

G is a bipartite graph in which, since $X \neq \emptyset$ and $Y \neq \emptyset$, $(V_1 - X) + (V_2 - Y)$ is a maximal biclique; $X \cup Y$ induces a stable set, so Theorem 2 follows from Characterization 2. □

Figure 2 illutrates Theorem 2 and Characterization 2.

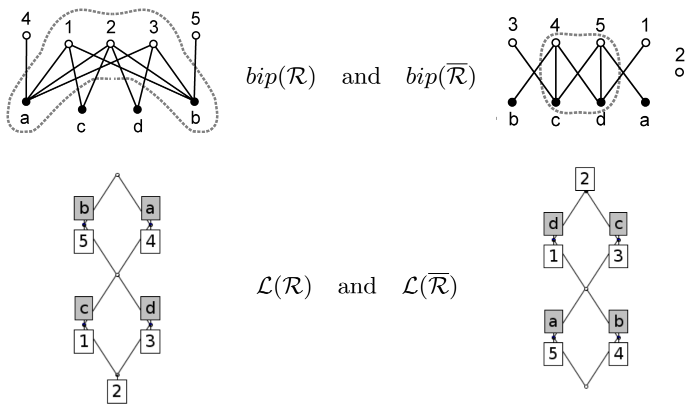


Fig. 2. Bipartite graph $bip(\mathcal{R})$ has a partition into maximal biclique $\{1, 2, 3, a, b\}$ and stable set $\{4, 5, c, d\}$; the corresponding lattice $\mathcal{L}(\mathcal{R})$ has an articulation point: $(123, ab)$. The mirror bipartite graph $bip(\overline{\mathcal{R}}) = mir(bip(\mathcal{R}))$ has also a partition (maximal biclique $\{4, 5, c, d\}$ and stable set $\{1, 2, 3, a, b\}$); the corresponding mirror lattice $\mathcal{L}(\overline{\mathcal{R}})$ has an articulation point: $(45, cd)$.

Let us remark that a similar class of bipartite graphs, called 'K+S' graphs was studied in [13], [14]. 'K+S' graphs are the bipartite graphs which can be partitioned into a maximal biclique and a stable set, and is thus a superclass of the bipartite graphs whose associated lattice has an articulation point: the reduced relation of a 'K+S' graph may correspond to a disconnected bipartite graph, so the corresponding lattice cannot have an articulation point. This is the case for instance for relation $\mathcal{R} = \{(1, a), (2, b), (3, a), (3, b)\}$.

3.2 Expressing the mirror articulation point

When concept (X, Y) is an articulation point of $\mathcal{L}(\mathcal{R})$, we could expect that $(\mathcal{O}-X, \mathcal{A}-Y)$ is an articulation point of $\mathcal{L}(\overline{\mathcal{R}})$. However, when an object x of X and/or a attribute y of Y fails to see vertices of the stable set $S = V-(X+Y)$, this is not exactly the expression of the mirror articulation point. The following theorem details the possible cases.

Theorem 3. *Let* (X, Y) *be an articulation point of* $\mathcal{L}(G)$*; then*

1. *If all the vertices of* X *and all the vertices of* Y *see* $S{=}V-(X+Y)$*, then* $(\mathcal{O}-X, \mathcal{A}-Y)$ *is an articulation point of* $\mathcal{L}(\overline{\mathcal{R}})$*.*

2. *If a set X' of objects of X (resp. $Y' \subset Y$) fail to see vertices of the stable set $S{=}V{-}(X{+}Y)$ but every attribute in Y (resp. object in X) sees S, then $((\mathcal{O}{-}X){+}X', \mathcal{A}{-}Y)$ [resp. $(\mathcal{O}{-}X, (\mathcal{A}{-}Y){+}Y')$] is an articulation point of $\mathcal{L}(\overline{\mathcal{R}})$.*
3. *If a set X' of objects of X and a set Y' of properties of Y fail to see vertices of the stable set $S = V{-}(X{+}Y)$, then $((\mathcal{O}{-}X){+}X', \mathcal{A}{-}Y)$ and $((\mathcal{O}{-}X), (\mathcal{A}{-}Y){+}Y')$ are two articulation points of $\mathcal{L}(\overline{\mathcal{R}})$.*

Proof:
Case 1: all the vertices of X and of Y see vertices of the stable set $S{=}V{-}(X{+}Y)$. $(\mathcal{O}{-}X, \mathcal{A}{-}Y)$ is an articulation point of $\mathcal{L}(\overline{\mathcal{R}})$: $\mathcal{O}{-}X \cup \mathcal{A}{-}Y$ is a stable set of $bip(\mathcal{R})$ by Characterization 2, so $(\mathcal{O}{-}X, \mathcal{A}{-}Y)$ is a biclique of $\mathcal{L}(\overline{\mathcal{R}})$. This biclique is maximal, since in $bip(\mathcal{R})$ no vertex of $(X{+}Y)$ fails to see $(\mathcal{O}{-}X, \mathcal{A}{-}Y)$.
Case 2: a set X' of objects of X fail to see vertices of the stable set $S{=}V{-}(X{+}Y)$ but every attribute in Y sees S. (Note that all the vertices of X' are equivalent, so if the relation is reduced, there is only one such object x'.) In this case, X' sees all the vertices of S in $mir(G)$, so $mir(G)((\mathcal{O}{-}X){+}(\mathcal{A}{-}Y))$ cannot be a maximal biclique; the corresponding maximal biclique will include X', so $((\mathcal{O}{-}X){+}X', \mathcal{A}{-}Y)$ is an articulation point of $\mathcal{L}(\overline{\mathcal{R}})$.
Naturally, the dual situation where a set of properties fails to see S is similar.
Case 3: Using the previous case, the existence of non-empty X' and Y' insure that $((\mathcal{O}{-}X){+}X', \mathcal{A}{-}Y)$ and $((\mathcal{O}{-}X), ((\mathcal{A}{-}Y){+}Y')$ are two distinct articulation points of $\mathcal{L}(\overline{\mathcal{R}})$. □

Remark 1. Conversely, there may be two consecutive articulation points of $\mathcal{L}(\mathcal{R})$ which correspond to a single one in the mirror lattice. In this case, both are irreducible elements of $\mathcal{L}(\mathcal{R})$.

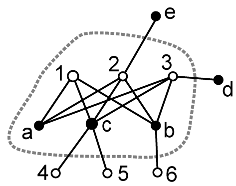
Figure 3 gives an example of a relation corresponding to Case 3 of Theorem 3.

3.3 Impact on the co-bipartite graph

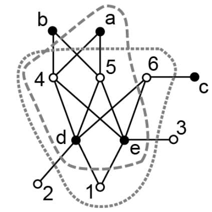
[4] showed that an articulation point of the lattice corresponds to a clique minimal separator of the co-bipartite graph. Let us remark that the converse does not hold: when there is a set X of universal vertices in bipartite graph $bip(\mathcal{R})$, for instance $X \subset \mathcal{O}$, then the neighborhood $N(X)$ of X in co-bipartite graph $cobip(\mathcal{R})$ is a clique separator, separating X from $\mathcal{A}$. The mirror co-bipartite has the same clique separator.

Property 1. Let $\mathcal{R}$ be a relation and $G{=}cobip(\mathcal{R})$ be its associated co-bipartite graph. Then $\mathcal{L}(\mathcal{R})$ has an articulation point (X, Y) if and only if G has a clique minimal separator $S = (\mathcal{O}{-}X){+}(\mathcal{A}{-}Y)$, minimally separating X from Y.

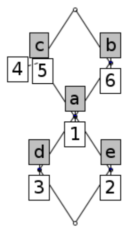
Corollary 1. *A co-bipartite graph has a clique minimal separator if and only if its mirror co-bipartite graph has a clique minimal separator.*



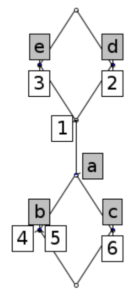
a) $G{=}bip(\mathcal{R})$



b) $bip(\overline{\mathcal{R}})$



c) $\mathcal{L}(\mathcal{R})$



d) $\mathcal{L}(\overline{\mathcal{R}})$

Fig. 3. a) A bipartite graph with partition into maximal biclique $\{1, 2, 3, a, b, c\}$ and stable set $\{4, 5, 6, d, e\}$ where 1 and a see only the biclique. **b)** The mirror bipartite graph. **c)**, **d)** The corresponding lattices, where $\mathcal{L}(\mathcal{R})$ has 1 articulation point but $\mathcal{L}(\overline{\mathcal{R}})$ has 2 corresponding articulation points.

3.4 Artificially creating an articulation point of the lattice

As described in [4], given a relation $\mathcal{R}$, an articulation point in lattice $\mathcal{L}(\mathcal{R})$ can be created by choosing a concept (X, Y), and saturating the corresponding minimal separator $S = (\mathcal{O}{-}X) \cup (\mathcal{A}{-}Y)$ of $cobip(\mathcal{R})$. This means that we modify relation $\mathcal{R}$ by *removing* any crosses from $\mathcal{R}(\mathcal{O} - X, \mathcal{A} - Y)$, obtaining relation $\mathcal{R}'$. This causes articulation point (X, Y) to appear in $\mathcal{L}(\mathcal{R}')$. A concept has disappeared from $\mathcal{L}(\mathcal{R})$ if and only if it is incomparable with concept (X, Y) in $\mathcal{L}(\mathcal{R})$. Similarly, a minimal separator S' disappears from the set of minimal separators of $cobip(\mathcal{R})$ if and only if S' crosses minimal separator S in $cobip(\mathcal{R})$.

Note that in the mirror relation $\overline{\mathcal{R}}$, crosses are *added* to create an articulation point of $\mathcal{L}(\mathcal{R}')$; $\mathcal{L}(\overline{\mathcal{R}})$ is also reorganized, but in a less straightforward fashion.

Figure 4 illustrates what happens when a concept is forced into an articulation point.

4 Finding the articulation points of a lattice

If a concept (X, Y) is an articulation point of $\mathcal{L}(G)$, then (X, Y) appears on any maximal chain of $\mathcal{L}(G)$. Thus we will first compute a maximal chain, and then use it to determine efficiently which concepts are articulation points.

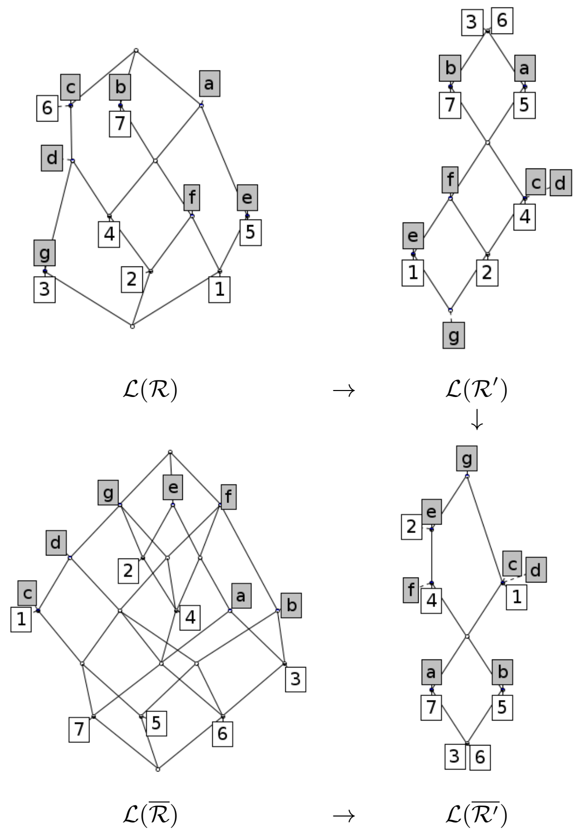


Fig. 4. A lattice on relation $\mathcal{R}$, the sublattice on relation $\mathcal{R}'$ obtained by forcing concept $(124, ab)$ of $\mathcal{L}(\mathcal{R})$ into an articulation point by removing crosses of $\mathcal{R}$, the corresponding mirror lattices.

4.1 Computing a maximal chain of the lattice

A maximal chain of the lattice can be computed using the sequence of degrees in a binary relation. We give an algorithm which repeatedly: finds an object x of maximum degree, whose intent $Y = \mathcal{R}(x)$ will belong to a concept (X, Y) covering the bottom element; removes $\mathcal{A} - Y$; uses the universal objects of the obtained subrelation to define the extent X of Y; and then removes X to compute the next concept of the maximal chain in the new relation, which is Bordat's subrelation for concept (X, Y). This corresponds to the process outlined in [4] to compute a maximal chain in $O(|\mathcal{O} + \mathcal{A}|.|\mathcal{R}|)$ time, for which we present a more efficient algorithm Max-Chain.

Theorem 4. *Algorithm* Max-Chain *computes a maximal chain of a lattice* $\mathcal{L}(\mathcal{R})$ *in* $O(min(|\mathcal{R}|, |\overline{\mathcal{R}}|))$ *time.*

ALGORITHM MAX-CHAIN
Input : A context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$
Output: A maximal chain $\mathscr{C}$ of $\mathcal{L}(\mathcal{R})$
prefix $\leftarrow \emptyset$; $\mathscr{C} \leftarrow \emptyset$;
repeat
  Choose an object x of maximum degree; $X \leftarrow \{x\}$;
  $Y \leftarrow \mathcal{R}(x)$;
  remove x and $\mathcal{A} - Y$ from $\mathcal{R}$;
  $U \leftarrow$ set of universal vertices of $\mathcal{R}$;
  $X \leftarrow X + U$;
  remove all vertices of U from $\mathcal{R}$;
  add concept $(prefix + X, Y)$ to $\mathscr{C}$;
  prefix $\leftarrow prefix + X$;
until $\mathcal{R}$ *is empty*;

Proof: Let x be an object of maximum degree in $\mathcal{R}$, then $\forall y \in \mathcal{O}$, $\mathcal{R}(x) \not\subset \mathcal{R}(y)$ [4]; x, with its equivalent objects forming set X, yields the extent of concept (X, Y) covering the bottom element, with $Y = \mathcal{R}(X)$ [4]. Bordat's subrelation, in which (X, Y) will correspond to the bottom concept, is then computed, by removing x and $\mathcal{A}-Y$ and finding the universal vertices, which will be the vertices which were in the same maximal clique module as x. The next computed element will be an atom of the new lattice.

Computing and ordering the degrees requires $O(|\mathcal{R}|)$ time. If a correct data structure is used (an adjacency list of $bip(\mathcal{R})$ linked to an ordered list of the degrees), the sequence of degrees can be updated in $O(1)$ time for each removal of crosses to form the new subrelation, so the overall cost for these updates costs $O(|\mathcal{R}|)$ time.

All these steps can be done in $\overline{\mathcal{R}}$ equivalently, so the overall time required is $O(min(|\mathcal{R}|, |\overline{\mathcal{R}}|))$. □

Example 1. Let us consider the lattice from Figure 5. Let us choose 1, which is of maximum degree; $\mathcal{R}(1) = \{a, b, c, d, e, f\}$; concept $(1, abcdef)$ is tentatively created. 1 and g are removed from the relation: there is no universal vertex, so $(1, abcdef)$ is a concept.

In the new relation, 2 is of maximum degree (5); $X \leftarrow 2$, $Y \leftarrow abcde$; we remove 2 and f from the relation; there is no universal vertex, so the next concept on our maximal chain will be $(12, abcde)$.

3 is now of maximum degree; $X \leftarrow 3$, $Y \leftarrow abce$; 3 and d are removed from the relation; 4 is now universal, so $X \leftarrow 34$; 4 is removed from the relation; concept $(1234, abce)$ is created.

5 is now of maximum degree; $X \leftarrow 5$, $Y \leftarrow abc$; 5 and e are removed from the relation; the new relation has no universal vertex, so the next concept is $(12345, abc)$.

6 is now of maximum degree; $X \leftarrow 6$, $Y \leftarrow bc$; 6 and a are removed and the relation becomes empty; the last concept is $(123456, bc)$.

We have generated maximal chain: $((1, abcdef), (12, abcde), (1234, abce), (12345, abc), (123456, bc))$.

	a	b	c	d	e	f	g	degree
1	×	×	×	×	×	×		6
2	×	×	×	×	×		×	6
3	×	×	×		×			4
4	×	×	×		×			4
5	×	×	×	×				4
6		×	×					2
7	×							1

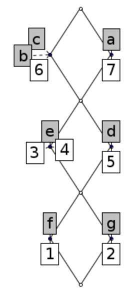


Fig. 5. A lattice with several articulation points. Example 1 computes maximal chain $((1, abcdef), (12, abcde), (1234, abce), (12345, abc), (123456, bc))$.

4.2 Computing the articulation points from a maximal chain of the lattice

Once we have obtained a maximal chain $((X_1, Y_1), (X_2, Y_2), \dots, (X_k, Y_k))$, we want to test each concept (X_i, Y_i) as to whether it is an articulation point. If it is, then by Theorem 2, $S = (\mathcal{O} - X) \cup (\mathcal{A} - Y)$ will be empty, as S is a clique separator of *cobip*$(\mathcal{R})$. Thus we will need to test for emptiness $\mathcal{R}(\mathcal{O}{-}X_1, \mathcal{A}{-}Y_1)$, $\mathcal{R}(\mathcal{O}{-}X_2, \mathcal{A}{-}Y_2)$, $\dots$, $\mathcal{R}(\mathcal{O}{-}X_k, \mathcal{A}{-}Y_k)$. Since $Y_k \subset \dots Y_2 \subset Y_1$, we need only to test $\mathcal{R}(\mathcal{O}{-}X_1, \mathcal{A}{-}Y_1)$, $\mathcal{R}(\mathcal{O}{-}X_2, Y_1{-}Y_2)$, $\dots$, $\mathcal{R}(\mathcal{O}{-}X_k, Y_{k-1}{-}Y_k)$.

Algorithm Articulations computes the set of articulation points of a lattice.

ALGORITHM ARTICULATIONS
Input : A context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$, a maximal chain $((X_1, Y_1), \dots, (X_k, Y_k))$ of $\mathcal{L}(\mathcal{R})$.
Output: Set $\mathscr{M}$ of articulation points of $\mathcal{L}(\mathcal{R})$
$\mathscr{M} \leftarrow \emptyset$;
for *i=1* **to***k* **do**
  $X \leftarrow \mathcal{O} - X_i$;
  if *i=1* **then**
    $Y \leftarrow \mathcal{A} - Y_1$;
  else
    $Y \leftarrow (Y_{i-1} - Y_i)$;
  if $X \times Y = \emptyset$ **then**
    $\mathscr{M} \leftarrow \mathscr{M} + (X_i, Y_i)$;

In Example 1, for maximal chain: $((1, abcdef), (12, abcde), (1234, abce), (12345, abc), (123456, bc))$, we will test: $\mathcal{R}(\{2,3,4,5,6,7\}, \{g\})$, $\mathcal{R}(\{3,4,5,6,7\}, \{f\})$, $\mathcal{R}(\{5,6,7\}, \{d\})$, $\mathcal{R}(\{6,7\}, \{e\})$, $\mathcal{R}(\{7\}, \{a\})$. We will find $(2, g) \in \mathcal{R}$, so $(1, abcdef)$ is not an articulation point, $\mathcal{R}(\{34567\}, \{f\}) = \emptyset$, so $(12, abcde)$ is an articulation point, $(5, d) \in \mathcal{R}$, so $(1234, abce)$ is not an articulation point, $\mathcal{R}(\{67\}, \{e\}) = \emptyset$, so $(12345, abc)$ is an articulation point, and finally $(7, a) \in \mathcal{R}$, so $(123456, bc)$ is not an articulation point.

4.3 Finding the clique minimal separator decomposition of a co-bipartite graph

Finding the articulation points of a lattice is equivalent to finding the clique minimal separators of the corresponding co-bipartite graph. Thus we can use Algorithm ARTICULATIONS to compute the clique minimal separators of a co-bipartite graph efficiently, and also easily extract the 'atoms' of the decomposition by clique minimal separators, and the mirror 'atoms'. This requires $O(min(|\mathcal{R}|, |\overline{\mathcal{R}}|))$ time to compute, *i.e.* less than the number of edges of the co-bipartite graph, since only the external edges are traversed.

Theorem 5. *Let $G = (\mathcal{O} + \mathcal{A}, E)$ be a co-bipartite graph on cliques $\mathcal{O}$ and $\mathcal{A}$, let $\mathcal{R}$ and $bip(\mathcal{R})$ be the corresponding bipartite graph and relation on $\mathcal{O} + \mathcal{A}$ (where $xy \in E$ if and only if $xy \notin \mathcal{R}$). Algorithm* ARTICULATIONS *returns an ordered set of articulation points of $\mathcal{L}(\mathcal{R})$, call it $((X_1, Y_1), \ldots, (X_k, Y_k))$. The clique minimal separators of G are $S_1 = (\mathcal{O} - X_1) \cup (\mathcal{A} - Y_1), \ldots, S_k = (\mathcal{O} - X_k) \cup (\mathcal{A} - Y_k)$. The corresponding atoms by clique minimal separator decomposition are: $T_1 = \mathcal{O} \cup (\mathcal{A} - Y_1), T_2 = (\mathcal{O} - Y_1) \cup (\mathcal{A} - Y_2), \ldots, T_{k+1} = (\mathcal{O} - Y_1) \cup \mathcal{A}$.*

In Example 1, with maximal chain: ($(1, abcdef)$, $(12, abcde)$, $(1234, abce)$, $(12345, abc)$, $(123456, bc)$), the articulation points are: $(12, abcde)$ and $(12345, abc)$. The clique minimal separators of $cobip(\mathcal{R})$ are: $\{3, 4, 5, 6, 7, f, g\}$ and $\{6, 7, d, e, f, g\}$ the atoms by clique minimal separator decomposition of $cobip(\mathcal{R})$ will be: $\mathcal{O} \cup \mathcal{A} - \{a, b, c, d, e\}$, $\mathcal{O} - \{1, 2\} \cup \mathcal{A} - \{a, b, c\}$ and $\mathcal{O} - \{1, 2, 3, 4, 5\} \cup \mathcal{A}$, so the atoms obtained by clique minimal separator decomposition are: $\{1, 2, 3, 4, 5, 6, 7, f, g\}$, $\{3, 4, 5, 6, 7, d, e, f, g\}$ and $\{6, 7, a, b, c, d, e, f, g\}$.

5 Lattices where every concept is an articulation point

When every concept is an articulation point, the lattice is just one maximal chain, which we call a chain lattice. In this case, the relation is a 'Guttmann scale': by ordering the elements by decreasing degree, a full triangular matrix is obtained.

5.1 Chain lattices and the corresponding graphs

By Theorem 2, every articulation point of $\mathcal{L}(\mathcal{R})$ corresponds to (at least one) articulation point of $\mathcal{L}(\overline{\mathcal{R}})$, so the following holds:

Property 2. $\mathcal{L}(\mathcal{R})$ is a chain lattice if and only if $\mathcal{L}(\overline{\mathcal{R}})$ is a chain lattice.

In view of the discussion from Subsection 3.2, the chain and the mirror chain do not necessarily have the same number of elements, although one can not be more than twice the length of the other. When $\mathcal{L}(\mathcal{R})$ is a chain lattice, in $cobip(\mathcal{R})$ all the minimal separators are clique separators, so the co-bipartite graph is chordal.

Theorem 6. *A co-bipartite graph is chordal if and only if its mirror co-bipartite graph is chordal.*

Since *cobip*$(\mathcal{R})$ is chordal, it has no C_4, so *bip*$(\mathcal{R})$ has no $2K_2$; such a bipartite graph is called a 'chain graph'. Our results give an alternate proof of the result from [18]:

Property 3. A bipartite graph G is a chain graph if and only if $mir(G)$ is a chain graph.

5.2 Recognizing chain lattices and the corresponding graphs

We will now see that we can test, in the same $O(min\{|\mathcal{R}|, |\overline{\mathcal{R}}|\})$ time as Algorithm MAX-CHAIN, the three equivalent properties:

- whether $\mathcal{L}(\mathcal{R})$ is a chain lattice;
- whether *bip*$(\mathcal{R})$ is a chain graph;
- whether *cobip*$(\mathcal{R})$ is chordal.

Given a context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$, we can efficiently recognize whether $\mathcal{L}(\mathcal{R})$ is a chain, using the results from Section 4: while computing a maximal chain $((X_1, Y_1), \ldots, (X_k, Y_k))$ of the lattice, add a counter which keeps track of the number of crosses of $\mathcal{R}$ involved; in the end, the lattice is a chain if and only if the counter's value is exactly $|\mathcal{R}|$. This is the same as testing whether $|X_1|.|Y_1| + \Sigma_{i=2}^{k} (|X_i| - |X_{i-1}|) . |Y_i| = |\mathcal{R}|$.

Recently, many graph recognition algorithms endeavour to add a 'certificate' to the answer; a certificate provides the user with a structure which is easy to verify and which enables to quickly check that the answer is indeed correct. In the case of chain graphs, for example, a recent result gives a certifying algorithm [15].

For chain graphs, we can provide a negative certificate in the form of an extraneous element $(x, z) \in \mathcal{R}$ found in $\mathcal{R}(\mathcal{O} - X_i, Y_{i-1} - Y_i)$ which prevents $\mathcal{L}(\mathcal{R})$ from being a chain. In this case, (x, z) corresponds to the lowest concept (X_i, Y_i) which is not an articulation point of $\mathcal{L}(\mathcal{R})$, with $x \in X_i$ and $z \notin Y_i$. In the lattice, there will be at least one concept which is not comparable with (X_i, Y_i), for instance the concept whose intent is $\mathcal{R}(x)$. In a similar fashion, $S = (\mathcal{O} - X_i) \cup (\mathcal{A} - Y_i)$ will be a non-clique minimal separator of *cobip*$(\mathcal{R})$, as edge xz is missing, certifying that *cobip*$(\mathcal{R})$ fails to be chordal. Finally, $\forall y' \in (Y_i - \mathcal{R}(x)), \forall x' \in (X_i - X_{i-1})$, $\{x, x', y, y'\}$ induces a $2K_2$ in *bip*$(\mathcal{R})$, a certificate that *bip*$(\mathcal{R})$ fails to be a chain graph.

5.3 Creating a chain lattice and corresponding graph embeddings

We will now examine what happens when we restrict lattice $\mathcal{L}(\mathcal{R})$ to one of its maximal chains. To do this, we will compute a maximal chain, $((X_1, Y_1), (X_2, Y_2), \ldots, (X_k, Y_k))$, as discussed in Subsection 4.1. We will then remove all crosses which do not correspond to this chain, i.e. we will need to empty

$\mathcal{R}(\mathcal{O}-X_1, \mathcal{A}-Y_1), \mathcal{R}(\mathcal{O}-X_2, \mathcal{A}-Y_2), \ldots, \mathcal{R}(\mathcal{O}-X_k, \mathcal{A}-Y_k)$, which as discussed before is equivalent to emptying $\mathcal{R}(\mathcal{O}-X_1, \mathcal{A}-Y_1), \mathcal{R}(\mathcal{O}-X_2, Y_1-Y_2), \ldots, \mathcal{R}(\mathcal{O}-X_i, Y_{i-1}-Y_i), \ldots, \mathcal{R}(\mathcal{O}-X_k, Y_{k-1}-Y_k)$.

As before, this can be done in $O(min(|R|, |\overline{R}|))$ time.
In Example 1 and the lattice from Figure 5, with maximal chain $((1, abcdef), (12, abcde), (1234, abce), (12345, abc), (123456, bc))$, relation $\mathcal{R}$ will be restricted to $\mathcal{R}'$, which can be re-organized into a triangular matrix:

$\mathcal{R}'$	b	c	a	e	d	f	g
1	×	×	×	×	×	×	
2	×	×	×	×	×		
3	×	×	×	×			
4	×	×	×	×			
5	×	×	×				
6	×	×					
7							

Since all the minimal separators of $cobip(\mathcal{R})$ have been saturated by restricting $\mathcal{R}$ to chain $\mathcal{R}'$, a minimal triangulation of $cobip(\mathcal{R})$ is thereby computed, by adding to $cobip(\mathcal{R})$ any missing edge from the Cartesian product $(\mathcal{O}-X_1)\times(\mathcal{A}-Y_1), (\mathcal{O}-X_2)\times(\mathcal{A}-Y_2), \ldots, (\mathcal{O}-X_k)\times(\mathcal{A}-Y_k)$. An existing algorithm [16] computes a minimal triangulation of a claw-free AT-free graph in linear time; co-bipartite graphs are claw-free AT-free graphs [4]; however, the above process can be considered as an improvement on the linear-time, since only the external edges of the co-bipartite are counted in the complexity analysis, but not the edges which lie inside cliques on $\mathcal{O}$ and $\mathcal{A}$.

Since the computed triangulation of $cobip(\mathcal{R})$ is minimal, we can ensure that we have removed an inclusion-minimal set of crosses from $\mathcal{R}$ to obtain a chain lattice; we also have removed an inclusion-minimal set of edges from $bip(\mathcal{R})$ to reduce it to a chain graph.

When examining what happens in the mirror relation $\overline{\mathcal{R}}$, we see that we have computed in $O(min(|R|, |\overline{R}|)$ time:

- a maximal sub-triangulation of the mirror co-bipartite graph $cobip(\overline{\mathcal{R}})$, for which the best known algorithm was the general one in $O(nm)$ time [2].
- a minimal embedding of $bip(\overline{\mathcal{R}})$ into a chain graph.

6 Conclusion and perspectives

In this paper, we investigate a property which is true in the relation and in the complement relation (which we call the mirror relation). This leads us to present linear time algorithms for both lattice and graph problems, such as computing a maximal chain of the lattice and computing a minimal triangulation of a co-bipartite graph.

When an articulation point is artificially created in a lattice $\mathcal{L}(\mathcal{R})$ by removing crosses from $\mathcal{R}$, we do not know exactly what happens to the mirror lattice $\mathcal{L}(\overline{\mathcal{R}})$, which is a strangely distorted image of $\mathcal{L}(\mathcal{R})$. We conjecture that the number of concepts decreases. The set of concepts may be contracted in a fashion which is exploitable, yielding more information than $\mathcal{L}(\mathcal{R})$, where a set of concepts is simply removed.

We also leave open the question of how the Galois subhierarchy is impacted by these transformations of the relation.

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