

# Consecutive-ones: handling lattice planarity efficiently<sup>3</sup>

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**Abstract.** A concept lattice may have a size exponential in the number of objects it models. Polynomial-size lattices and/or compact representations are thus desirable. This is the case for planar concept lattices, which has both polynomial size and representation without edge crossing, but a generic process for drawing them efficiently is yet to be found. Recently, it has been shown that when the relation has the consecutive-ones property (i.e, the matrix of the relation can be rapidly reordered so that the 1s are consecutive in every row), the number of concepts is polynomial and these can be efficiently generated. In this paper we show that a consecutive-ones relation  $|\mathcal{R}|$  has a planar lattice which can be drawn in  $O(|\mathcal{R}|)$  time. We also give a hierarchical classification of polynomial-size lattices based on structural properties of the relation  $\mathcal{R}$ , its associated graphs  $G_{bip}$  and  $G_{\mathcal{R}}$ , and its concept lattice  $\mathcal{L}(\mathcal{R})$ .<sup>3</sup>

**Keywords:** *consecutive-ones matrix, consecutive-ones relation, planar lattice, polynomial lattice, chordal-bipartite graph, Ferrers dimension.*

## 1 Introduction

There is a strong relationship between concept lattices and graphs, which enables to use the rich mine of graph results. For example, [4] presented a very efficient algorithm to generate the concepts when the relation has the consecutive-ones property. They used the natural association between a finite context  $(\mathcal{O}, \mathcal{P}, \mathcal{R})$  and a bipartite graph  $G_{bip} = (\mathcal{O} + \mathcal{P}, E)$ , where  $xy \in E$  iff  $(x, y) \in \mathcal{R}$ . Another interesting feature is that these consecutive-ones relation have a few  $O(|\mathcal{R}|)$  concepts, and can be recognized and reorganized in very efficient  $O(|\mathcal{R}|)$  time ([8]). This relationship was also illustrated by [5] who defined an encoding graph  $G_{\mathcal{R}}$  which is the complement of  $G_{bip}$  (i.e.,  $G_{\mathcal{R}} = (\mathcal{O} + \mathcal{P}, F)$ ,  $\mathcal{O}$  and  $\mathcal{P}$  are cliques and  $\forall x \in \mathcal{O}, \forall y \in \mathcal{P}, xy \in F$  iff  $(x, y) \notin \mathcal{R}$ ).  $G_{\mathcal{R}}$  was then used by [3] for generating all the concepts in the general case.

In [6], the problem of restricting a relation to a relation which has a polynomial number of concepts was addressed by suggesting to embed the relation

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into a relation whose graph  $G_{\mathcal{R}}$  is weakly chordal. In this case, the lattices  $\mathcal{L}(\mathcal{R})$  have  $O(|\mathcal{R}|)$  concepts.

To avoid handling exponential lattices, a  $O(|\mathcal{O} + \mathcal{P}|)$  size substructure called Galois sub-hierarchy has been defined. There are several algorithms that compute the Galois sub-hierarchy in at least  $O(|\mathcal{O} + \mathcal{P}| \cdot |\mathcal{P}|^2)$  (see [1]).

Recently, [20] discussed about a characterization of the planar lattices as having a Ferrers dimension at most to two (see Theorem 1 below). Also concerned with the number of concepts, he pointed out that these lattices have at most  $O(|\mathcal{P}^3|)$  concepts. However, actually obtaining a planar drawing remains difficult.

Graph results enables us to remark that relations with Ferrers dimension at most two are a supcase of consecutive-ones relations, and a subcase of relations for which graphs  $G_{bip}$  are weakly chordal. This gives the idea that there is an interesting hierarchy of classes of concept lattices which need to be researched.

In this paper, we address two closely related problems:

- We discuss how to obtain an efficient planar drawing of the lattice for the sub-class of planar lattices whose relations have the consecutive-ones property.
- We study a hierarchical classification of polyomial size concept lattices, and give precise bibliographical references. This hierarchy combines results based on graph theory, order and lattice theory, and matrix patterning that should be helpful for further work on lattice representations.

## 2 Background

An undirected graph  $G = (V, E)$  is said to be *chordal* (or *triangulated*) if it has no chordless cycle of length greater than 3. A graph  $G$  is said to be *weakly chordal* if it and its complement,  $\bar{G}$ , has no chordless cycle of length greater than 4. A *bipartite graph* is a graph  $G = (V_1 + V_2, E)$ , where  $V_1$  and  $V_2$  are independent sets (i.e., each induces an edgeless subgraph). A *chordal-bipartite* graph is a graph that is bipartite and weakly chordal. The *neighborhood* of a vertex  $v$  in a graph  $G = (V, E)$  is denoted and defined as  $N(v) = \{x \in V | vx \in E\}$ . A bipartite graph is a *chain graph* if for each  $V_i$   $i \in \{1, 2\}$ , the neighborhoods of vertices of  $V_i$  can be totally ordered by set containment (i.e., for any pair of vertices  $u, v \in V_i$ , either  $N(u) \subseteq N(v)$  or  $N(v) \subseteq N(u)$ ; equivalently, the graph has no induced  $2K_2$ ).<sup>4</sup>

A relation  $\mathcal{R} \subseteq (\mathcal{O} \times \mathcal{P})$  is said to be a *Ferrers relation* if  $\forall x_1, x_2 \in \mathcal{O}, \forall y_1, y_2 \in \mathcal{P}, (x_1, y_1) \in \mathcal{R}$  and  $(x_2, y_2) \in \mathcal{R}$  implies  $(x_1, y_2) \in \mathcal{R}$  or  $(x_2, y_1) \in \mathcal{R}$ . The *Ferrers dimension* of a relation  $\mathcal{R}$  is the minimum number of Ferrers relations the intersection of which is  $\mathcal{R}$ . The *chain dimension* of a bipartite graph is the minimum number of chain graphs the intersection of which yields the graph. Therefore, the Ferrers dimension of a relation  $\mathcal{R}$  is equal to the chain dimension of  $G_{bip}$ . A *chain graph* is a graph with chain dimension 1; thus  $\mathcal{R}$  is a Ferrers relation iff  $G_{bip}$  is a chain graph.

<sup>4</sup> A  $2K_2$  of an undirected graph  $G$  is a quadruple of vertices  $x, y, z, t$  such that  $xy$  and  $zt$  are the only edges of  $G$  whose endpoints both are in  $\{x, y, z, t\}$ .

Planar lattices are characterized by their Ferrers dimension:

**Theorem 1.** ([2],[7],[11]) *The following are equivalent:*

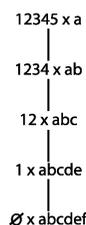
- The Ferrers dimension of  $\mathcal{R}$  is at most 2;
- $\mathcal{L}(\mathcal{R})$  has a planar representation;
- The order dimension of  $\mathcal{L}(\mathcal{R})$  is at most 2;
- $\mathcal{L}(\mathcal{R})$  has a conjugate order.

When  $G_{\mathcal{R}}$  is chordal ([5]), the corresponding matrix of a  $\mathcal{R}$  presents a very specific pattern, called a *staircase*, as the rows are totally ordered by inclusion (see an example in Figure 1). We then have:

**Theorem 2.** *The following are equivalent:*

- $\mathcal{R}$  is a Ferrer relation;
- $\mathcal{R}$  has a staircase matrix;
- $G_{bip}$  is a chain graph;
- $G_{\mathcal{R}}$  is chordal;
- $L(\mathcal{R})$  is a chain.

	a	b	c	d	e	f
1	x	x	x	x	x	
2	x	x	x			
3	x	x				
4	x	x				
5	x					



**Fig. 1.** A staircase matrix of a relation, and its concept lattice which is a chain.

### 3 Consecutive-ones lattices

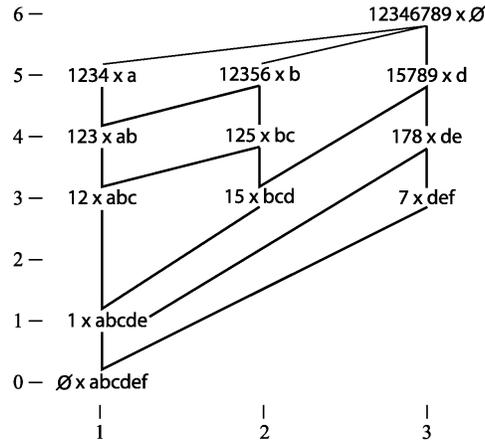
**Consecutive-ones lattices are planar.** A relation is said to have the *consecutive-ones property (for rows)* if the columns of its binary matrix can be ordered such that in each row all the ones are consecutive. Figure 2 shows such a relation and the corresponding concept lattice. Planarity of consecutive-ones lattices is a direct consequence of Theorem 1 applied to the following observation.

**Observation 3** *If a relation has the consecutive-ones property, then its Ferrers dimension is at most 2.*

**Proof:** Let  $M$  be a consecutive-ones matrix of a relation  $\mathcal{R}$ . Let  $\mathcal{F}_1$  be the relation obtained from  $M$  by: for each row, changing to one each zero that occurs before the sequence of ones in the row. Let  $\mathcal{F}_2$  be the relation obtained from  $M$  by: for each row, changing to one each zero that occurs after the sequence of ones in the row. By Theorem 2,  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are Ferrers relations, and clearly  $\mathcal{R} = \mathcal{F}_1 \cap \mathcal{F}_2$ . Thus, the Ferrers dimension of  $\mathcal{R}$  is at most 2.  $\diamond$

Note that Observation 3 is also implied by a characterization of a larger class of relations (see [15]), those of Ferrers dimension 2 for which  $\mathcal{F}_1 \cup \mathcal{F}_2$  is complete (which corresponds to the interval digraphs).

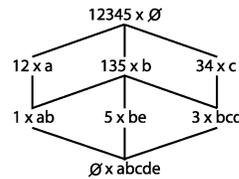
	a	b	c	d	e	f
1	×	×	×	×	×	
2	×	×	×			
3	×	×				
4	×					
5		×	×	×		
6		×				
7				×	×	×
8				×	×	
9				×		



**Fig. 2.** A consecutive-ones relation and the associated concept lattice. In the matrix, objects and properties are ordered such that in each row all the 1s are consecutive. The margin of the Lattice drawing gives the coordinates of the concepts, as computed by a execution of algorithm PLACOL (see below).

Not all planar lattices have the consecutive-ones property, as illustrated in Figure 3. In the matrix of this example the columns can not be reordered to obtain a consecutive-ones matrix: columns  $a$  and  $b$  must be consecutive (because of object 1) and so do columns  $b$ ,  $c$ , and  $d$  (because of object 3); thus no permutation of the columns can erase the hole on row 5.

	a	b	c	d	e
1	×	×			
2	×				
3		×	×	×	
4			×		
5		×			×



**Fig. 3.** A relation that does not have the consecutive-ones property, and the corresponding planar lattice.

Unfortunately, the Ferrers dimension does not give much insight towards an efficient generic process for generating a planar drawing. We now present an algorithm, which we call PLACOL (for *PLANar drawing of Consecutive-Ones Lattices*), that use the specificity of a consecutive-ones matrix to efficiently build a planar representation of the lattice. With a slight modification, this algorithm computes the Galois sub-hierarchy.

**The algorithm.** We use as input a consecutive-ones matrix in which the rows are ordered by increasing value of starting column SC. The rows sharing the same SC are ordered by decreasing value of ending column EC; thus, defining

one of the staircases  $s$  of the matrix. Note that several rows may have the same EC value and the same SC value. The algorithm constructs for each staircase, a chain of the lattice each concept of which is associated with a row of the matrix (see [4] for more explanation on this process). Tables HEAD and TOP memorize the concepts from which an edge has to be drawn respectively to a concept of a later chain (lateral edge) or to the top. The edges from the bottom to the atoms are drawn online. For example, for the matrix  $M$  of Figure 2, PLACoL first generates the bottom  $\emptyset \times \mathcal{P}$  as  $M[1, f] = 0$ . Staircase 1 (rows 1-4) successively generates the elements of the first chain  $1 \times abcde \sim 1234 \times a$ . Staircase 2 (rows 5-6) generates chain  $15 \times bcd \sim 12356 \times b$  and the lateral edges outgoing from the first chain. Staircase 3 (rows 7-9) generates chain  $7 \times def \sim \text{top}$  and the lateral edges outgoing from staircase 2. PLACoL ends by drawing the edges incoming to the top. The position of each concept is determined by its chain number (abscissa) and its intent size (ordinate) as described below.

#### Algorithm PLACoL

**Input:** A consecutive-ones matrix in the form described above.

**Output:** A planar drawing of the corresponding concept lattice.

**Process:**

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If the first row of the matrix ended with a zero then
    Generate bottom =  $\emptyset \times \mathcal{P}$ ;
    ABSCISSA(bottom)  $\leftarrow$  1; ORDINATE(bottom)  $\leftarrow$  0;
    Create an edge from bottom to the first concept generated next;
// else the first concept generated next will be the bottom.
For each staircase  $s$  do:
    Let  $t$  be the first row of  $s$ ;
    For each list  $R$  of rows of  $s$  sharing the same EC do:
        Let  $r$  be the last row of  $R$ ;
        If  $\exists i \in [t..r] \mid \text{SC}(i) = \text{SC}(s)$  then
            Let  $u$  be the first such row  $i$ ;
            If  $\text{SC}(r) < \text{SC}(s)$  then  $A \leftarrow \emptyset$  else  $A \leftarrow [u..r]$ ;
             $B \leftarrow [\text{SC}(s).. \text{EC}(r)]$ ;
            Create concept  $A \times B$ ;
            TOP( $s$ )  $\leftarrow A \times B$ ;
            ASBCISSA( $A \times B$ )  $\leftarrow s$ ;
            ORDINATE( $A \times B$ )  $\leftarrow |\mathcal{P}| - |A|$ ; // i.e., 0 or ( $r - u$ )
        If  $A \times B$  is not the first concept generated for staircase  $s$  then
            // Edge between consecutive concepts of same chain.
            Create an edge from the previously generated concept to  $A \times B$ ;
        If  $\exists i \in R \mid \text{SC}(i) < \text{SC}(r)$  then // Create a lateral edge.
            Let  $q$  be the last such row in  $R$ ,  $h$  the corresponding staircase;
            Create edge from HEAD( $q$ ) to  $A \times B$ ;
            TOP( $h$ )  $\leftarrow A \times B$ ;
    else if  $A \times B$  first concept of  $s$  and extent(bottom) empty then
        Create an edge from bottom to atom  $A \times B$ ;

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If  $s$  is not the last staircase and  $EC(r) \geq SC(s + 1)$  then  
   Insert rows  $R$  in staircase  $s + 1$ ;  
   // *preserving increasing order on EC, and on SC.*  
    $HEAD(r) \leftarrow A \times B$ ;  
 If  $r$  is the last row of the last staircase then  
   If the intent of the last generated concept is  $\mathcal{P}$   
     then this concept is the top element;  
     else generate the top concept  $\mathcal{O} \times \emptyset$ ;  
 For each staircase  $s$  do:  
   Create an edge from  $TOP(s)$  to the top element.

We first prove that Algorithm PLACOL actually generates the concepts in an order compatible with a planar drawing.

**Lemma 4** *Algorithm PLACOL constructs successive chains (first one with bottom, last one with top, each chain labeled by its generating staircase), which yields a partition of the lattice in which a concept is generated only after all its ancestors have been generated.*

**Proof:** The first staircase,  $s = 1$ , generates all the concepts that contain the first property  $SC(1)$  (corresponding to the first column). By Theorem 2, these concepts form a chain of concepts ordered by increasing size of extent  $[t..r]$ . By construction of the matrix, all the concepts generated afterwards will not have  $SC(1)$  in their intents, and thus, will not be ancestors of any concept of the first chain; thus the first generated chain is actually a chain of the lattice containing exactly those concepts with  $SC(1)$  in their intent. Recursively, each staircase  $s$  generates the concepts the intents of which contain  $SC(s)$  and not  $SC(1)$  through  $SC(s - 1)$ ; this corresponds to a chain in which each concept will not be an ancestor of a previously generated concept. As each concept is generated once (see [4]), this yields a partition of the lattice into chains; i.e., a linear extension which respects the claimed property on concepts.  $\diamond$

We now know that the only edges that can be created in Algorithm PLACOL will be edges going up from a given concept to another concept higher either in the same chain or laterally to a concept generated by a later staircase. There remains to prove that these lateral edges are non-crossing edges.

**Theorem 5.** *Assume an execution of Algorithm PLACOL on a consecutive-ones matrix, resulting in a corresponding concept lattice diagram. If the diagram has an edge from a concept  $C_s$  of generated chain  $s$  to a concept  $C_u$  of generated chain  $u \neq s$ , then  $u > s$  and this induces no crossing edge that is impossible to avoid.*

**Proof:** Let  $C_s$  and  $C_u$  be concepts respectively generated by two different staircases  $s$  and  $u$ . Suppose the diagram contains an edge  $e$  from  $C_s$  to  $C_u$ .

- By Lemma 4,  $s > u$  is impossible.
- If  $u = s + 1$ , there is no obstacle.

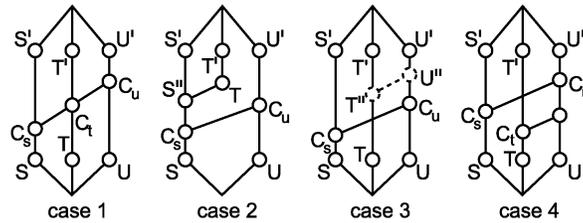
– If  $u > s + 1$  there will be at least one intermediate chain  $t$ . Let  $S$  be the first concept generated by chain  $s$  and  $S'$  its last concept. We similarly define  $T, T', U,$  and  $U'$ , for chains  $t$  and  $u$ . Four exclusive situations could occur, which are illustrated by Figure 4:

**Case 1.**  $e$  is a transitivity edge and therefore is not in the diagram of  $\mathcal{L}(\mathcal{R})$ . Otherwise,  $e$  is not a transitivity edge: next cases.

**Case 2.**  $T$  is a descendent of  $C_s$  – i.e., there is an edge  $S''-T$  with  $S''$  descendent of  $C_s$  in chain  $s$ . Thus edge  $e$  can be drawn below edge  $S''-T$  and there is no crossing. Otherwise,  $e$  is forced to cross some non-transitivity edge of an intermediate chain  $t$ : next cases.

**Case 3.**  $C_u$  is not a descendent of any concept of  $t$ . Note there may be some edge from a concept  $T''$  of  $t$  to a concept  $U''$  of  $u$  which is a descendent of  $C_u$ , but it does not matter. Since  $C_u$  is not a descendent of  $T$ , the rows of  $T$  (i.e., the last objects in its extent) have not been used to construct  $C_u$ , i.e.,  $\text{EC}(T) < \text{EC}(C_u)$ . As we are not in case 2,  $T$  is not a descendent of  $C_s$  (nor an ancestor) and then  $\text{EC}(T) > \text{EC}(C_s)$ . As a consequence  $\text{EC}(C_s) < \text{EC}(C_u)$  which contradicts the existence of the edge  $e$ .

**Case 4.** Otherwise,  $C_u$  is a descendent of some concept  $C_t$  of  $t$ . Let  $I = \text{intent}(C_s) \cap \text{intent}(C_t)$ . Since  $C_u$  is a descendent of both  $C_s$  and  $C_t$ , we have  $\text{intent}(C_u) \subset I \neq \emptyset$ . If  $I = \text{intent}(C_t)$ , then  $\text{intent}(C_t) \subset \text{intent}(C_s)$  and  $C_t$  is a descendent of  $C_s$ . Then  $e$  is a transitivity edge. If  $I \subset \text{intent}(C_t)$ , there must exist a concept  $X$  such that  $\text{intent}(X) = I$ ,  $X$  is a descendent of  $C_s$  and  $C_t$ , and  $X$  is on chain  $t$ . Since,  $\text{intent}(C_u) \subset I = \text{intent}(X)$ ,  $C_u$  is a descendent of  $X$ . Then  $e$  is a transitivity edge. This last case is thus impossible.  $\diamond$



**Fig. 4.** Different situations for the proof of Theorem 5.

We now prove that all the edges of the diagram are actually provided by Algorithm PLACOL. For this we need the following lemma:

**Lemma 6** *In the diagram of a consecutive-ones lattice, each concept different from top has at most one incoming edge from concepts of a previous chain generated by Algorithm PLACOL.*

**Proof:** Suppose the diagram has both edge  $C_s-C_u$  and edge  $C_t-C_u$ . If  $C_s, C_t,$  and  $C_u$  be concepts generated by three different staircases  $s \leq t \leq u$ . This corresponds exactly to the case number 4 of Theorem 5, and thus, is impossible.

If  $C_s$  and  $C_t$  are generated by the same staircase, and thus, are in the same chain. W.l.o.g.  $C_s$  is an ancestor of  $C_t$ . As a consequence, edge  $C_s-C_u$  is a transitivity edge.  $\diamond$

**Theorem 7.** *Algorithm PLACOL provides all the edges of the diagram.*

**Proof:** Recall, by Lemma 4, PLACOL partitions the concepts of the lattice in to a collection of chains. Suppose the diagram contains an edge  $e=C_1-C_2$ . Either  $C_1$  and  $C_2$  are in the same chain of this partition or not. Concepts  $C_1$  and  $C_2$  are generated by the same staircase iff they are in a same chain; then  $e$  is created by the algorithm iff  $C_1$  and  $C_2$  are consecutive. If  $C_1$  and  $C_2$  have been generated by two different staircases  $i$  and  $j$  respectively, then  $i < j$ , and this means that rows of staircase  $i$  have been inserted in the following staircases until (at least) staircase  $j$ . Algorithm PLACOL detects whether there exists such an insertion and, if yes, selects the most recent such staircase  $q$  (i.e., of highest index) and the corresponding concept, which is given by HEAD. By Lemma 6, such a situation occurs at most once for each concept and the algorithm creates a lateral edge. Finally, each highest concept of a chain will be linked to the top, unless this highest concept is an ancestor of a concept of another chain. TOP ensures there will be no created edge between top and a concept that has been proven to be a ancestor of some concept of a later chain. Thus, each edge of the diagram is generated exactly once.  $\diamond$

**Complexity analysis.** As a consecutive-ones lattice has at most  $O(|\mathcal{R}|)$  concepts ([4]) and, by Lemma 6, the diagram of a consecutive-ones concept lattice has at most  $O(|\mathcal{R}|)$  edges, each of them is generated without extra cost. Thus, Algorithm PLACOL has the same  $O(|\mathcal{R}|)$  complexity as Algorithm CONS-1.

**Drawing the diagram.** The above considerations do not ensure a correct planar drawing of the lattice, as some pair of edges that could be drawn without crossing may cross in an incautious drawing. Fortunately, this problem can be solved by choosing for each concept  $C$  an ordinate value  $y(C)$  that is a function of the size of its intent:  $y(C) = |\mathcal{P}| - |\text{intent}(C)|$ . As  $\text{intent}(C)$  is an interval, its size is computed in constant time. Note that the ordinates can be computed using the extent instead of the intent, especially when Algorithm PLACOL is modified to give the whole extent label (see below).

Furthermore, each edge is drawn when its ending point is reached and, at this time, its starting point is memorized in HEAD; there is thus no need to memorize all the previously generated concepts – an expensive constraint for many lattice generating algorithms.

The abscissa of a concept is given by the number of the chain to which this concept belongs. With this, the bottom's abscissa is 1 and the top's abscissa is the number of staircases in the input matrix. We may chose several values both for the initial abscissa and for the increment.

If we want to set bottom and top on the same vertical line, all the coordinates can be rotated accordingly, using a simple mathematical formula: if  $(x_0, y_0)$  are

the coordinates of bottom and  $(x_1, y_1)$  the coordinates of top, we will rotate all the coordinates by angle  $\theta = \text{ArcSin}((x_1 - x_0) / \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2})$ . Consequently, the previously verticality will have a new  $\frac{\pi}{2} + \theta$  direction; the other edges, which had a direction in interval  $]0, +\frac{\pi}{2}[$ , will have a direction in  $]\theta, \theta + \frac{\pi}{2}[$ , which remains correct because  $0 < \theta \leq +\frac{\pi}{2}$ . Other changes to coordinates may be chosen, as desirable for different purposes; in particular, for the respect of attribute-additivity (see [14]).

**Concept labeling.** In order to obtain a time complexity of  $O(|\mathcal{R}|)$ , Algorithm CONS-1 only labels the intent of each concept. A complete labeling of the extent would have an elementary cost corresponding to the size of each generated extent. This is due to the fact that a consecutive-ones matrix ensures each *intent* is an interval, but this is not true for the *extents*, as a concept created by a set  $R$  of rows of a staircase  $u$  can use rows of a previous staircase  $s$  without using any row of some intermediate staircase  $t$  (in other words, some row of  $s$  has an EC greater or equal to the EC of the rows in  $R$ , but no row of  $t$  does).

It is possible to memorize and update a list of the rows whose ECs are not outdated, and this can be done in  $O(|\mathcal{O}|)$  per concept. Consequently, the total complexity of Algorithm CONS-1, as well of Algorithm PLACoL, would be  $O(|\mathcal{O}| \cdot |\mathcal{R}|)$ , if we want a complete labeling of the extents.

On the other hand, for lattice drawing, the concepts need to be labeled only by the introduced objects and/or properties. Then, in Algorithm PLACoL we chose to label the intent with the introduced objects. Computation of a concept  $C$  is determined by the set  $R$  of rows defined within a given augmented staircase  $s$  by a common value of EC. This will determine the introduced objects of  $\text{intent}(C)$ .  $\text{Intent}(C)$  is partitioned into three sets. The first set corresponds to the rows that have been inserted in  $s$  and inherited from a previous staircase; they are characterized by a SC value different than  $\text{SC}(s)$ . The second set corresponds to the rows that have been used to create a previous concept of the same chain; they are characterized by a EC bigger than the smallest EC of  $R$ . The last set corresponds to the rows that have not been previously used. The objects introduced by  $C$  are exactly the ones of the third set, which are computed with no extra-cost by Algorithm PLACoL.

We can provide the corresponding introducer-labeling for the properties, making a few slight changes in Algorithm PLACoL; this will have no impact on the complexity. As all the rows of the same staircase share the same SC and are ordered by decreasing value of EC, only the last concept of a chain may introduce a property. As the concepts of a chain have no descendent in a previously generated chain, a concept cannot introduce a property that is in the intent of some concept of a chain that is generated later. Thus, the last concept of each chain  $s$  is the introducer of properties in interval  $[\text{SC}(s) .. \text{SC}(s + 1)[$ , except for the last chain  $l$  (in which the last concept introduces  $[\text{SC}(l) .. \text{EC}(l)]$ ). If  $z$  is the first column of the matrix with no 1, the properties of  $[z .. |\mathcal{P}|]$  are introduced by bottom. Each property  $p$  of the remaining interval  $]\text{EC}(l) .. z[$  is introduced by the last concept of the last chain whose row's EC is bigger than  $p$ ; this can be

determined in a lazy way by preprocessing the column that labels each staircase with the appropriate list of properties from  $\text{]EC}(l) \dots z[$ .

**Galois sub-hierarchy.** As a consequence, Algorithm PLACOL can also be modified to compute the Galois sub-hierarchy: using the introducer-labeling, all the concepts whose intent and extent are both empty will be dismissed. The remaining concepts are the elements of the sub-hierarchy. When a concept is dismissed, it is replaced by its highest maintained ancestor in the same chain for the outgoing edges and by its lowest maintained descendent in the same chain for the incoming edges; bottom and top are managed accordingly. Thus, using elementary techniques, the total complexity remains the same: the Galois sub-hierarchy is computed in  $O(|\mathcal{R}|)$  time, which is better than the generic algorithms ([1]), and should be refinable to  $O(|\mathcal{O} + \mathcal{P}|)$ .

Figure 5 give a planar drawing of the lattice of Figure 2 with the standard labeling of the concepts.

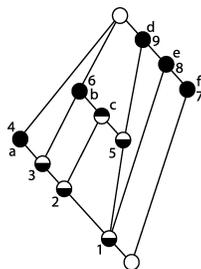


Fig. 5. A standard planar drawing of the lattice of Figure 2.

## 4 A hierarchy of polynomial concept lattices

In this section, we present a hierarchical classification of polynomial-size concept lattices using a known hierarchy of bipartite graph classes:  $\{\text{chain}\} \subset \{\text{biconvex}\} \subset \{\text{convex}\} \subset \{\text{ATE-free}\} \subset \{\text{chordal-bipartite}\} \subset \{\text{bipartite}\}$  (for more information on these graph classes, see [19], [9], and [10]<sup>5</sup>). This classification shows that studying the length of cycles in graphs ([12]) is important for concept lattices.

1. The smallest class corresponds to the lattices that are simply a chain, as in Figure 1. This corresponds to the situation of Theorem 1.
2. The second class corresponds to matrices that have the consecutive-ones property for the objects as well as for the properties.

<sup>5</sup> Note the use of ‘bichordal’ for chordal-bipartite, ‘bigraph’ for bipartite graph, ‘digraph’ for directed graph.

	Matrix $M$	Relation $\mathcal{R}$	Bipartite graph $G_{bip}$	Co-bip. graph $G_{\mathcal{R}}$	Lattice $\mathcal{L}(\mathcal{R})$
1	staircase	Ferrers dim. = 1	chain graph	chordal	chain, size $O( \mathcal{O} )$
2	doubly consecutive-ones		biconvex		
3	consecutive-ones		convex		size $O( \mathcal{R} )$
4	see [10]	Ferrers dim. $\leq 2$	chain dim. $\leq 2$	circular-arc	planar
5			ATE-free		
6	$\Gamma$ -free		chordal-bipartite	weakly chordal	
7					polynomial size

**Fig. 6.** The hierarchy of polynomial-size concept lattices. The smaller class is above.

3. A wider class corresponds to the consecutive-ones property; as shown above, the corresponding lattices are planar.  $G_{bip}$  is convex.
4. The next class is the class of planar lattices. They are characterized by Theorem 1, which is difficult to handle (see e.g., [20]). [10, 17, 18] indirectly give another characterization of planar lattices by setting an equivalence between a Ferrers dimension  $\leq 2$  relation and a pattern in the matrix. They also prove that the graphs  $G_{bip}$  in this case are exactly the interval containment bipartite graphs. In [17, 16], the graphs  $G_{\mathcal{R}}$  are characterized as circular-arc co-bipartite graphs.
5. The previous class of bipartite graphs is properly contained in a subclass of chordal-bipartite graphs, the class of ATE-free chordal-bipartite graphs (see [10]).
6. In [6], it is proved that a concept lattice for which  $G_{bip}$  is chordal-bipartite (equivalently, weakly chordal and bipartite) has  $O(|\mathcal{R}|)$  concepts. All the previous classes we present are included in this one, and thus, have polynomial-size concept lattices. Testing whether a graph is chordal-bipartite can be done in  $\min\{O(|\mathcal{O} + \mathcal{P}|^2), O(|\mathcal{R}| \cdot \log(|\mathcal{O} + \mathcal{P}|))\}$  time (see e.g., [19] for a discussion), and this can be performed directly on the matrix  $M$ . These graph algorithms compute a doubly lexical ordering of the matrix and search for a specific forbidden pattern called a  $\Gamma$ <sup>6</sup>. Every doubly lexical ordering has no  $\Gamma$  iff the corresponding graph is chordal-bipartite. We know how important is the use of total orders on objects or properties for concept generation: arbitrary order such as lexic (i.e. lexical) order of [13] or structural order as domination of [3].
7. Our largest class is the one of polynomial-size concept lattices.

## 5 Conclusion

We have demonstrated that the special structure of consecutive-ones relation matrices can be exploited to efficiently draw a planar concept lattice diagram. The hierarchy of polynomial-size lattices illuminates the relationship between several special classes of relations (alternatively, graphs) and concept lattices, including those lattices that are planar. Further study of these relationships is likely to yield new results in the efficient representation of concept lattices; in particular, for those lattices that have "almost planar" representations. In

<sup>6</sup> For  $h < i, j < k$ ,  $M[h, j] = M[h, k] = M[i, j] = 1$  and  $M[i, k] = 0$  forms a  $\Gamma$  of matrix  $M$ .

addition, one might study how to complete (or restrict) an arbitrary relation into a special class in order to exploit special structure by adding (or removing) a small (bounded) or inclusion-minimal set of ones. A chordal-bipartite graph has no chordless cycle of size  $\geq 6$ ; it would be interesting to study classes of bipartite graphs  $G_k$  that permit chordless cycles of increasing size (i.e., a graph in the class  $G_k$  has no chordless cycle of size  $\geq k$ ) to see if the corresponding concept lattices remain polynomial sized and whether the concepts can be efficiently generated. Ultimately, one would like to characterize the class of polynomial-size concept lattices.

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