Representing a concept lattice by a graph

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Abstract

Concept lattices (also called Galois lattices) are an ordering of the maximal rectangles defined by a binary relation. In this paper, we present a new relationship between lattices and graphs: given a binary relation R, we define an underlying graph G_R, and establish a one-to-one correspondence between the set of elements of the concept lattice of R and the set of minimal separators of G_R.

We explain how to use the properties of minimal separators to define a sublattice, decompose a binary relation, and generate the elements of the lattice.

Key words: concept lattice, Galois lattice, co-bipartite graph, minimal separator.

1 Introduction

One of the important challenges in data handling is generating or navigating the concept lattice of a binary relation.

Concept lattices are well-studied as a classification tool ([1]), are used in several areas of Database Managing, such as Object-Oriented Databases ([44]), inheritance lattices ([22,10]), mining for association rules ([45,33]), generating frequent sets ([43]), and are a promising aid for the rapidly emerging field of data mining for biological databases.

In this paper, we present a new paradigm for describing and understanding concept lattices, by equating the concepts of the lattice with the set of minimal separators of an underlying graph.

The notion of minimal separator, introduced by Dirac in 1961 to characterize chordal graphs ([13]), has been extensively studied during the past decade on

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non-chordal graphs ([25,24,31,3,42]), and has yielded many new theoretical and algorithmical graph results.

We apply some of these results to analyzing and decomposing a binary relation and the associated lattice. The mechanisms involved are illustrated on a running example.

The paper is organized as follows:

Section 2 gives some preliminary notions on concept lattices and graph separators, and presents our example. For undefined notions, the reader is referred to the classical works of [11] and [17]. In Section 3, we define the underlying graph G_R which we use to represent a binary relation R, describe some of its properties, and explain how it relates to the concept lattice $\mathcal{L}(R)$. In Section 4, we define a sublattice by making into a clique a minimal separator of the underlying graph. Section 5 shows how to use a clique minimal separator to decompose a binary relation. In Section 6, we compute the successors of an element. Section 7 addresses the algorithmic issue of generating all the elements of the lattice efficiently.

2 Preliminaries

2.1 Concept lattices

Originally, the lattice defined by a binary relation R was known as the Galois lattice of R, as described by Barbut and Monjardet ([1]), and was studied by several mathematicians. Later, Ganter and Wille ([16]) introduced the wider notion of 'context', renamed these lattices as 'concept lattices', and studied them extensively, with many interesting results. When the terminologies between these two tendencies differ, we will give both terms in the definitions below.

Given a finite set $\mathcal{P}$ of "properties" or "attributes" (which we will denote by lowercase letters) and a finite set $\mathcal{O}$ of "objects" or "tuples" (which we will denote by numbers), a binary relation R is a proper subset of the Cartesian product $\mathcal{P} \times \mathcal{O}$; the triple $(\mathcal{P}, \mathcal{O}, R)$ is called a **context**. We will refer to the elements of the relation as **ones**, and to the non-elements as **zeroes**. Given a subset $\mathcal{P}'$ of $\mathcal{P}$ and a subset $\mathcal{O}'$ of $\mathcal{O}$, we will say that the set $R \cap (\mathcal{P}' \times \mathcal{O}')$ is a **sub-relation** of R.

Definition 2.1 *Given a context $C = (\mathcal{P}, \mathcal{O}, R)$, a* ***concept*** *or* ***closed set*** *of C, also called a* ***maximal rectangle*** *of R, is a sub-product $A \times B \subseteq R$ such* *that* $\forall x \in \mathcal{O} - B, \exists y \in A \mid (y, x) \notin R$, *and* $\forall x \in \mathcal{P} - A, \exists y \in B \mid (x, y) \notin R$. *A is called the* ***intent*** *of the concept, B is called the* ***extent****.*

Note that in general, a context will define an exponential number of concepts.

Example 2.2 $\mathcal{P} = \{a, b, c, d, e, f\}$, $\mathcal{O} = \{1, 2, 3, 4, 5, 6\}$. *The table below describes binary relation R:*

	a	b	c	d	e	f
1		×	×	×	×	
2	×	×	×			
3	×	×				×
4				×	×	
5			×	×		
6	×					

$bc \times 12$ *and* $abf \times 3$ *are maximal rectangles (concepts) of R. bc is the* ***intent*** *of rectangle* $bc \times 12$, *and* 12 *its* ***extent****.*

A **lattice** is a partially ordered set in which every pair $\{A, A'\}$ of elements has both a lowest upper bound (denoted by **join**(A, A')) and a greatest lower bound, (denoted by **meet**(A, A'), [11]), extending the notion of lowest common ancestor for a pair of nodes in a tree.

Given a context $C = (\mathcal{P}, \mathcal{O}, R)$, the concepts of C, ordered by inclusion on the intents, define a lattice, called a **concept lattice** or **Galois lattice**. A dual lattice is defined by inclusion on the extents. We represent a lattice by the Hasse diagram of the partial ordering on all maximal rectangles: transitivity and reflexivity arcs are omitted. Concepts are often referred to as **elements** of this lattice.

Such a lattice, sometimes referred to as a complete lattice, has a smallest element, called the **bottom element**, and a greatest element, called the **top element**.

An element $A' \times B'$ is said to be a **descendant** of element $A \times B$ if $A \subset A'$. An element $A' \times B'$ is said to be a **successor** of element $A \times B$ if $A \subset A'$ and there is no intermediate element $A'' \times B''$ such that $A \subset A'' \subset A'$. The set of successors of an element is called the **cover** of this element. The successors of the bottom element are called **atoms**.

The notions of **predecessor**, **ancestor** and **co-atom** are defined dually.

A path from bottom to top in the Hasse diagram is called a **maximal chain** of the lattice.

Example 2.3 *Figure 1 gives the concept lattice of the relation of our example.* $ab \times 23$ *and* $bc \times 12$ *are not comparable,* $ab \times 23$ *is a successor of* $a \times 236$, $a \times 236$ *is a predecessor of* $ab \times 23$*. The atoms of* $\mathcal{L}(R)$ *are:* $a \times 236$, $b \times 123$, $c \times 125$ *and* $d \times 145$*.* $(\emptyset \times 123456$, $b \times 123$, $bc \times 12$, $abc \times 2$, $abcdef \times \emptyset)$ *is a maximal chain of the lattice.*

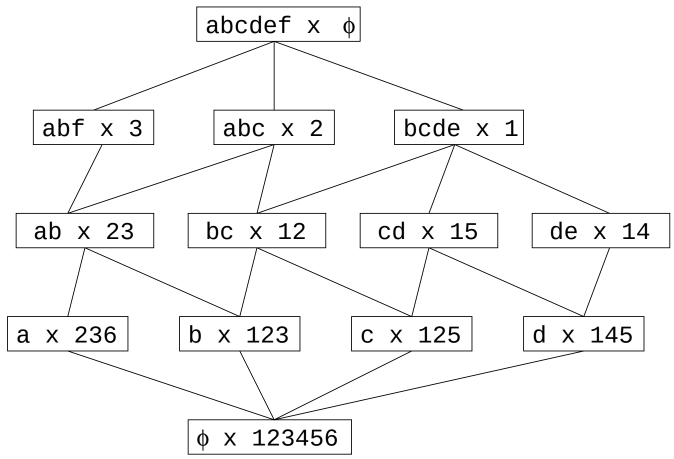


Figure 1. Concept lattice $\mathcal{L}(R)$ of relation R of Example 2.2.

2.2 Graphs

The graphs used in this work are finite and undirected. A graph is denoted $G = (V, E)$; V is the vertex set, $|V| = n$ and $E \subseteq V^2 = \{\{x, y\} | x, y \in V, x \neq y\}$ is the edge set, $|E| = m$. For $X \subset V$, $G(X)$ denotes the subgraph induced by X in G. The **neighborhood** of vertex x (the set of vertices y such that xy is an edge of E) is denoted by $N(x)$. If xy is an edge of E, we say that x and y **see** each other. For $X \subset V$, $N(X) = \bigcup_{x \in X}(N(x)) - X$. A **clique** is a set X of pairwise adjacent vertices (i.e. $\forall x \neq y \in X, xy \in E$). An **independent set** (sometimes called a stable set) is a set X of pairwise non-adjacent vertices (i.e. $\forall x \neq y \in X, xy \notin E$).

An **asteroidal triple** of vertices ([27]) is an independent set of three vertices $\{x_1, x_2, x_3\}$ such that for every pair (x_i, x_j) of vertices of this triple, there is a path from x_i to x_j which does not intersect $N(x_k)$, where x_k is the third vertex of the triple. A graph is said to be **AT-free** if it has no asteroidal triple of vertices. A **claw** is a subgraph isomorphic to $K_{1,3}$, a graph on four vertices x_1, x_2, x_3, x_4 with only edges x_1x_2, x_1x_3 and x_1x_4.

We will also need the notion of minimal triangulation, which is the process

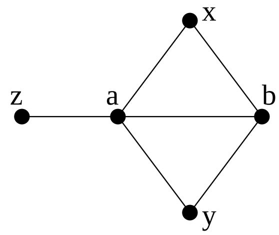


Figure 2. Graph of example 2.5.

of embedding a graph into a chordal graph by the addition of an inclusion-minimal set of edges. A graph is said to be **chordal** (or triangulated) if it contains no chordless induced cycle of length strictly greater than three.

Definition 2.4 *([36]) Let $G = (V, E)$ be a non-chordal graph; $H = (V, E+F)$ is a* ***minimal triangulation*** *of G if H is chordal and for all proper subset F' of F, graph $(V, E + F')$ fails to be chordal.*

The basic notion we use in this work is that of minimal separator.

A **separator** S of a connected graph G is a subset of vertices such that subgraph $G(V - S)$ is disconnected. S is called an **xy-separator** if x and y lie in different connected components of $G(V - S)$; S is called a **minimal xy-separator** if S is an xy-separator and no proper subset of S separates x from y. Finally, S is called a **minimal separator** if there is some pair $\{x, y\}$ of vertices such that S is a minimal xy-separator. Note that if $xy \notin E$, then the graph has at least one minimal xy-separator.

Example 2.5 *In the graph from Figure 2, $S = \{a, b\}$ is an xy-separator and an yz-separator. $S' = \{a\}$ is also an yz-separator. S is a minimal xy-separator, but not a minimal yz-separator, since S contains a smaller yz-separator S'.*

The following characterization is often used in graph papers:

Property 2.6 *Let $G = (V, E)$ be a connected graph, let S be a vertex set. S is a minimal separator of G iff there are at least two distinct connected components A and B of $G(V - S)$ such that $N(A) = N(B) = S$; A and B are called* full components.

A separator S is called a **clique separator** if it is a separator and a clique; we will say that we **saturate** a non-clique separator S if we add all missing edges necessary to make S into a clique.

A vertex is said to be **universal** if it sees all the other vertices of the graph.

Property 2.7 *A vertex is universal iff it belongs to* ***all*** *the minimal separators* *of the graph.*

Proof: Let x be a universal vertex of a graph $G = (V, E)$; suppose there is some minimal separator S which does not contain x; let C be the component of $G(V - S)$ which x belongs to, let C' be a second component of $G(V - S)$, let y be a vertex of C': clearly, $xy \notin E$, which contradicts the assumption that x is universal in G.

Conversely, Let x be a vertex which is not universal; let y be a vertex which x does not see: there must be a minimal separator which separates x from y and therefore does not contain x. □

As a consequence of Property 2.7, if X is the set of universal vertices of graph G, and S is a non-empty set of vertices, then S is a minimal separator of a connected component of $G(V - X)$ iff $S \cup X$ is a minimal separator of G. The set of universal vertices of a graph can be found in linear ($O(m)$) time.

Definition 2.8 *A subset X of vertices is said to be a **clique module** iff $\forall x, y \in X, \{x\} \cup N(x) = \{y\} \cup N(y)$.*

Belonging to a maximal clique module defines an equivalence relation ([4]), and it is easy to show that the corresponding partition can be computed in linear time using Hsu and Ma's partition refinement algorithm ([23]), which is described on chordal graphs, but works just as well on arbitrary graphs. In the rest of this paper, we will often refer to a maximal clique module X as if it was a vertex, with degree $|N(X)|$.

Definition 2.9 *A vertex x is simplicial if $N(x)$ is a clique, a maximal clique module X is simplicial if $N(X)$ is a clique.*

Simplicial vertices can be seen as the 'opposite' of universal ones, as illustrated by the following property, which is the mirror of Property 2.7:

Property 2.10 *([3]) A vertex is simplicial iff it belongs to **no** minimal separator of the graph.*

We will discuss simplicial vertices again in Section 3.

3 The co-bipartite graph underlying a binary relation

In a previous work [5], it is shown that the elements of the Galois lattice of the incidence relation of an undirected graph define separators of the complement of the graph. This leads us to represent a given context by a graph constructed on the complement of the relation.

Definition 3.1 *Let $C = (\mathcal{P}, \mathcal{O}, R)$ be a context; we will define an associated underlying graph, denoted G_R, as follows:*

- *The vertex set of G_R is $\mathcal{P} \cup \mathcal{O}$.*
- *$\mathcal{P}$ and $\mathcal{O}$ are cliques.*
- *For a vertex x of $\mathcal{P}$ and a vertex y of $\mathcal{O}$, there is an xy edge in G_R iff (x, y) is **not** in R.*

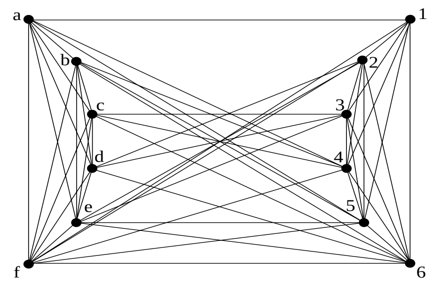


Figure 3. Underlying graph G_R of relation R of Example 2.2.

Note that only the edges between a vertex of $\mathcal{P}$ and a vertex of $\mathcal{O}$ are relevant and need be traversed when searching the graph; thus m will refer to $|\mathcal{P} \cup \mathcal{O}| - |R|$. In order to make our illustrations clearer, we will omit the internal edges of $\mathcal{P}$ and $\mathcal{O}$ in our figures in the rest of this paper.

By construction, the graph G_R we have just described belongs to the class of co-bipartite graphs. The graphs of this class have several remarkable properties, such as being AT-free and claw-free. This class is also hereditary: any subgraph of a co-bipartite graph which has more than one vertex is again co-bipartite. Moreover, since the relations we work on are considered as non-empty, graph G_R is always connected.

This ensures several nice properties on the minimal separators of co-bipartite graphs, which makes them easier to handle than on more general graphs.

Lemma 3.2 *An independent set in a co-bipartite graph is of size at most two.*

Proof: Suppose there exists a co-bipartite graph $G = (V, E)$ with an independent set $X \subseteq V$ of size three or more. By definition of a co-bipartite graph, V can be partitioned into two cliques. As X contains at least three vertices, at least two of them are in the same clique, a contradiction. $\square$

Corollary 3.3 *A co-bipartite graph is AT-free and claw-free.*

Corollary 3.4 *Let G be a co-bipartite graph constructed on cliques $\mathcal{P}$ and $\mathcal{O}$; then every minimal separator S of G has exactly 2 connected components,*

A and B, the first of which contains only vertices of $\mathcal{P}$ and the second only vertices of $\mathcal{O}$.

We can also give a characterization for the minimal separators in co-bipartite graphs, derived from Property 2.6:

Characterization 3.5 *Let S be a vertex set of a co-bipartite graph $G = (V, E)$; S is a minimal separator of G iff $G(V - S)$ has exactly two connected components A and B such that $N(A) = N(B) = S$.*

We are now ready to prove our main result:

Main Theorem 3.6 *Let $C = (\mathcal{P}, \mathcal{O}, R)$ be a context, let $G_R = (V, E)$ be the corresponding co-bipartite graph, let $A \neq \emptyset \subset \mathcal{P}$, $B \neq \emptyset \subset \mathcal{O}$; then $A \times B$ is a concept of R iff $S = V - (A \cup B)$ is a minimal separator of G_R.*

Proof: Let $C = (\mathcal{P}, \mathcal{O}, R)$ be a context, $G_R = (V, E)$ the corresponding co-bipartite graph.

(1) Let $A \times B$ be a concept of R, with $A \neq \emptyset$, $B \neq \emptyset$, and $A \cup B \neq V$, let $S = V - (A \cup B)$ $S = V - (A \cup B)$ is not empty. We claim that for each $a \in A, b \in B$, S is a minimal ab-separator of G_R. First of all, S is an ab-separator: if there was an edge ab in G_R, then by definition (a, b) would not be in R and therefore $A \times B$ would not be a concept. Next we will prove that S is minimal: suppose that it is not; by Property 2.6, w.l.o.g. there must be some vertex $x \in S$ such that x sees no vertex of B, which means that $\forall y \in B, (x, y) \in R$; $Ax \times B$ would be a rectangle of R, which contradicts the minimality of $A \times B$.

(2) Conversely, let S be a minimal separator of G_R, let A and B be the connected components of $G(V - S)$. As connected components, A and B are not empty; as a separator, S is not empty, then $A \cup B \neq V$. Since $\forall x \in A$, $\forall y \in B$, $xy \notin E$ and thus $(x, y) \in R$, we can conclude that $A \times B$ is a rectangle of R. Suppose $A \times B$ fails to be maximal: w.l.o.g. $\exists x \in \mathcal{O} - B$, $\forall y \in A$, $(y, x) \in R$. Thus $x \in S$ and, in G_R, x will see no vertex of A, so by Property 2.6, S fails to be minimal, a contradiction.

□

Definition 3.7 *Let $A \times B$ be a concept of relation R, let $S = V - (A \cup B)$. We will say that minimal separator S* ***represents*** *concept $A \times B$.*

We can now reformulate Characterization 3.5 to show that, given only the intent or the extent of a concept, it is easy to infer both parts of the concept:

Characterization 3.8 *Let $A \times B$ be a rectangle of relation R, with $A \neq \emptyset$,*

$B \neq \emptyset$, and $A \cup B \neq \mathcal{P} \cup \mathcal{O}$; then $A \times B$ is a concept iff in G_R, $N(A) = N(B)$.

Main Theorem 3.6 endows the minimal separators of G_R with a lattice structure. This structure is related to the lattice structure of the so-called minimal *ab*-separators of a graph shown by Escalante in [14], which we will mention again in Section 7, and also to the lattice structure of subsets of vertices described by [21,37,34,19,5].

From Main Theorem 3.6, we can deduce that a co-bipartite graph may have an exponential number of minimal separators, since a concept lattice can have an exponential number of elements. It is well known that, for a given size of $\mathcal{P}$, the largest lattice obtainable is the lattice describing all the subsets of $\mathcal{P}$, and that the corresponding relation has exactly one zero in each column and exactly one zero in each line (in this case, of course, $|\mathcal{P}| = |\mathcal{O}|$). The corresponding co-bipartite graph with a maximum number of minimal separators is thus the graph in which $|\mathcal{P}| = |\mathcal{O}|$ and each vertex of $\mathcal{P}$ sees exactly one vertex of $\mathcal{O}$.

Example 3.9 *In Figure 4, $S = \{a, d, e, f, 3, 4, 5, 6\}$ is a minimal separator of graph G_R of Figure 3, separating $C_1 = \{b, c\}$ from $C_2 = \{1, 2\}$, and $bc \times 12$ is a concept of R and an element of $\mathcal{L}(R)$. In G_R, $N(\{b, c\}) = N(\{1, 2\}) = S$.*

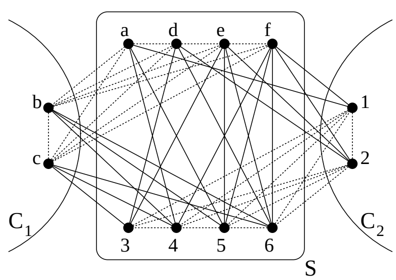


Figure 4. Separator $S = \{a, d, e, f, 3, 4, 5, 6\}$ of G_R.

We will now discuss how special cases such as lines of zeroes or ones of the relation, or lattice notions such as join and meet operations and atoms and co-atoms, can be interpreted in terms of graphs.

Interpretation of the lines of zeroes and lines of ones of the relation

Any line (column or row) of **zeroes** of a binary relation R corresponds to a universal vertex of G_R. Thus, according to Property 2.7, this line can be deleted from R without modifying the set of concepts of $(\mathcal{P}, \mathcal{O}, R)$, and can likewise be deleted from G_R without modifying the structure of the minimal separators of the graph. We will use this remark in Section 5 when decomposing a relation.

If R contains a line x of **ones**, then x will be simplicial in G_R and thus, by Property 2.10 will belong to no minimal separator of G_R. Therefore, it will appear in every concept of $(\mathcal{P}, \mathcal{O}, R)$ and can be removed from R to decrease the number of edges during computation of the concepts.

Join and meet operations

It is easy, given two minimal separators of G_R, to find the join and meet of the concepts that they represent.

Property 3.10 *Let $A_1 \times B_1$, $A_2 \times B_2$ be two elements of the lattice. Let $S_1 = V - (A_1 \cup B_1)$, let $S_2 = V - (A_2 \cup B_2)$, let $Y = S_1 \cup S_2$, let $J = (V - Y) \cap \mathcal{O}$ and $M = (V - Y) \cap \mathcal{P}$; then J is the extent of Join($A_1 \times B_1$, $A_2 \times B_2$) and M is the intent of Meet($A_1 \times B_1$, $A_2 \times B_2$).*

This can be deduced from the following property:

Property 3.11 *([11]) Let $A_1 \times B_1$, $A_2 \times B_2$ be two elements of a concept lattice. Then $B_1 \cup B_2$ is the extent of Join($A_1 \times B_1$, $A_2 \times B_2$) and $A_1 \cup A_2$ is the intent of Meet($A_1 \times B_1$, $A_2 \times B_2$).*

Atoms and co-atoms

In [4], the notion of moplex was introduced as a general definition of the extremity of a graph. It is interesting to note that the moplexes of the underlying graph G_R correspond precisely to the non-trivial extremities of the lattices: its atoms and co-atoms.

Definition 3.12 *([4]) A vertex set X which defines a maximal clique module, and such that $N(X)$ is a minimal separator, is called a **moplex**.*

Property 3.13 *Let R be a relation with no lines of ones, let $\mathcal{L}(R)$ be the corresponding concept lattice, let G_R be the underlying graph. If $A \times B$ is an atom of $\mathcal{L}(R)$ then A is a moplex of G_R; if $A \times B$ is a co-atom of $\mathcal{L}(R)$, then B is a moplex of G_R; there are no other moplexes in G_R.*

Proof: Let $C = (\mathcal{P}, \mathcal{O}, R)$ be a context, $\mathcal{L}(R)$ the corresponding concept lattice, and $G_R = (V, E)$ the corresponding co-bipartite graph.

(1) Let $A \times B$ be an atom of $\mathcal{L}(R)$, represented by minimal separator $S = N(A) = N(B)$. Clearly, as A is a subset of $\mathcal{P}$, it is a clique. We claim that A is a module: suppose that it is not. By Definition 2.8, there must exist x, y in A such that $N(x) \neq N(y)$. We can suppose w.l.o.g. that there exists a vertex b in $\mathcal{O}$ such that $xb \in E$ and $yb \notin E$. As a consequence, $A \times B$ has a predecessor, the intent of which is $A' \subseteq A - \{x\}$ and the extent of which is $B' \supseteq B \cup \{b\}$. Moreover, as $y \in A'$ and R has no lines of ones, $A' \times B'$ cannot be the bottom element $\emptyset \times \mathcal{O}$ of R. Therefore, $A \times B$

fails to be an atom, a contradiction. We can conclude, by Definition 3.12, that A is a moplex.

A similar proof shows dually that if $A \times B$ is a co-atom of $\mathcal{L}(R)$ then B is a moplex.

(2) Conversely, let A be a moplex of G_R, and let $S = N(A)$ be the associated minimal separator. By Characterization 3.5, S defines two connected components, one of which is A; the other will be denoted B, with $N(B) = S$. Suppose $A \subseteq \mathcal{P}$; by Characterization 3.8, $A \times B$ is a concept of $\mathcal{L}(R)$. As $A \neq \emptyset$ and R has no lines of zeroes, then $A \times B$ is not the bottom element of $\mathcal{L}(R)$ and has thus at least one predecessor. As A is a module, for all $x, y \in A$, $N(x) = N(y)$. As a consequence, the only way of extending B in order to have a predecessor of $A \times B$ in the lattice is to remove all vertices of A, which can only result in $\emptyset \times \mathcal{O}$, the bottom element of $\mathcal{L}(R)$. Thus, $A \times B$ is an atom of $\mathcal{L}(R)$.
If $A \subseteq \mathcal{O}$, we prove dually that $B \times A$ is a co-atom of $\mathcal{L}(R)$.

□

4 Selecting a sublattice by saturating a minimal separator

Computing a minimal triangulation of a graph is an important problem, with many applications.

Recent work has shown that minimal separators could be used to compute a minimal triangulation, essentially by repeatedly saturating a minimal separator of the graph ([25,32,2]). The process of saturating one minimal separator causes a number of other minimal separators to disappear from the graph; this process was first introduced by [25], and is extensively studied in [32] and [31] and its mechanism is described and used in [7].

In this Section, we will examine what happens to the lattice when a minimal separator of the underlying graph is saturated.

Definition 4.1 *([25]) Let S and T be two minimal separators of graph G; T is said to* **cross** *S if there are two different connected components C_1 and C_2 of $G(V - S)$ such that $T \cap C_1 \neq \emptyset$ and $T \cap C_2 \neq \emptyset$.*

Theorem 4.2 *([31]) A minimal separator of a graph G is a clique separator iff it does not cross any other minimal separator of G.*

Property 4.3 *([32]) Let G be a graph, let S be a minimal separator of G, let G_S denote the graph obtained from G by saturating S; then T is a minimal* *separator of G_S iff T is a minimal separator of G and T does not cross S in G.*

We will use this result on our underlying graph G_R: saturating a minimal separator S of G_R defines a new relation R', in which for each xy edge added to S, the corresponding element (x, y) is deleted from R. According to Property 4.3, we expect every concept of the resulting relation R' to be a concept of the original relation R.

Theorem 4.4 *Let R be a binary relation, and G_R the corresponding underlying co-bipartite graph. Let S be a minimal separator of G_R, representing concept $A \times B$ in lattice $\mathcal{L}(R)$, let R' be the new relation obtained by saturating S. Then the following two properties hold:*

(1) Concept lattice $\mathcal{L}(R')$ can be obtained from $\mathcal{L}(R)$ by removing all the elements which are not comparable to $A \times B$.
(2) Concept lattice $\mathcal{L}(R')$ is a sublattice of the original lattice $\mathcal{L}(R)$.

To prove this, we will need the following Lemma, which establishes the relationship between non-crossing minimal separators in a graph and comparable elements in a lattice.

Lemma 4.5 *Let R be a binary relation, let $\mathcal{L}(R)$ be the associated concept lattice, and let G_R be the corresponding underlying co-bipartite graph. Let S and S' be minimal separators of G_R, let $A \times B$ and $A' \times B'$ respectively be the concepts which S and S' represent; then S and S' are non-crossing minimal separators of G_R iff $A \times B$ and $A' \times B'$ are comparable elements in $\mathcal{L}(R)$.*

Proof: Let S and S' be two minimal separators, respectively representing concepts $A \times B$ and $A' \times B'$.

(1) Suppose S and S' are non-crossing. By Definition 4.1, this implies w.l.o.g. that $S \cap A' \neq \emptyset$. Then $A' \subseteq (A \cup B)$ and, as $A' \subseteq \mathcal{P}$ and $B \subseteq \mathcal{O}$, $A' \subseteq A$. Thus, $A \times B$ is a descendant of $A' \times B'$; these concepts are therefore comparable.
(2) Suppose S and S' are crossing. By Definition 4.1, $S \cap A' \neq \emptyset$ and $S' \cap A \neq \emptyset$; then $A' \not\subseteq A$ and $A \not\subseteq A'$. As a consequence, concepts $A \times B$ and $A' \times B'$ are not comparable.

□

Proof: (of Theorem 4.4) Let R be a relation, $\mathcal{L}(R)$ its concept lattice, G_R the underlying graph, and S a minimal separator of G_R. Let R' be the relation obtained from R by saturating S and $\mathcal{L}(R')$ its concept lattice. By Property 4.3, saturating S causes to disappear from the graph exactly those minimal separators which are non-crossing with S. Thus, by Lemma 4.5, concepts which are not comparable with $A \times B$ disappear from $\mathcal{L}(R)$. As a result, $\mathcal{L}(R')$ is a sublattice of $\mathcal{L}(R)$. $\square$

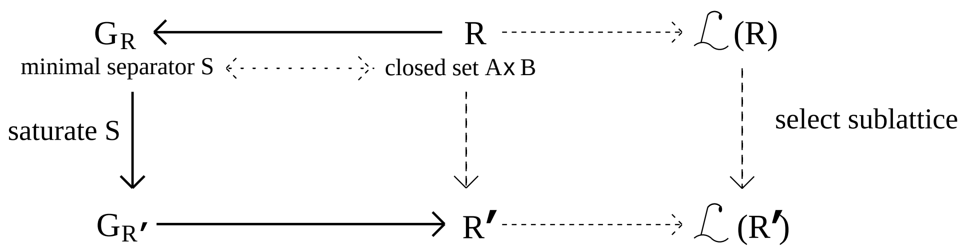


Figure 5. Relationships between relation, graph and lattice while saturating a separator.

Theorem 4.4 defines a process which enables us to restrict a binary relation R to a sub-relation $R' \subset R$ such that $\mathcal{L}(R')$ is a sublattice of $\mathcal{L}(R)$. This may prove important in many applications, as arbitrarily restricting a relation will not, in general, yield a sublattice, and can even cause the resulting lattice to be larger than the original one; indeed, not much is known on the exact mechanisms which govern the number of concepts defined by a given binary relation.

Example 4.6 *Let us saturate separator* $S = \{a, d, e, f, 3, 4, 5, 6\}$ *of* G_R *in Figure 4, representing concept* $bc \times 12$*. Edges* $a3$*,* $a6$*,* $d4$*,* $d5$*,* $e4$ *and* $f3$ *will be added.*

Figure 6 gives the new relation R' *obtained. Figure 7 gives the sublattice* $\mathcal{L}(R')$ *obtained. Saturating* S *has caused concepts* $a \times 236$*,* $ab \times 23$*,* $abf \times 3$*,* $d \times 145$*,* $cd \times 15$ *and* $de \times 14$ *to disappear from the lattice.*

We will conclude this section by discussing the minimal triangulations of G_R, as related to the minimal separator saturation process.

We will first remark that by virtue of the results in [29] and [31] on AT-free and claw-free graphs, all the minimal triangulations of G_R are proper interval graphs. (The reader is referred to [27], [35] and [17] for the definitions of interval graphs and proper interval graphs).

Property 4.7 *([3]) Given an input graph* G*, the process of repeatedly choosing a minimal separator of* G *which is not a clique, and saturating it, until all minimal separators are cliques, yields a minimal triangulation of the input graph in less than* n *steps. Moreover, this process characterizes the minimal triangulations of* G*: each minimal triangulation* H *of* G *is characterized by the minimal separators of* H*.*

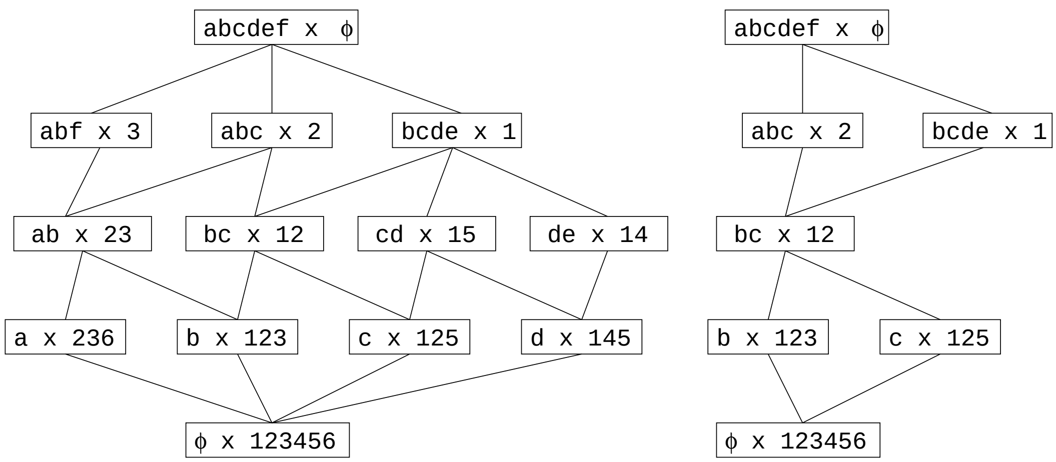
	a	b	c	d	e	f
1		×	×	×	×	
2	×	×	×			
3	×	×				×
4				×	×	
5			×	×		
6	×					

R

	a	b	c	d	e	f
1		×	×	×	×	
2	×	×	×			
3		×				
4						
5			×			
6						

R'

Figure 6. Original relation R; new relation R' defined by saturating minimal separator $S = \{a, d, e, f, 3, 4, 5, 6\}$.



$\mathcal{L}(R)$ $\mathcal{L}(R')$

Figure 7. Original lattice $\mathcal{L}(R)$; lattice $\mathcal{L}(R')$ obtained by saturating the minimal separator which represents concept $bc \times 12$.

Property 4.8 *Computing a minimal triangulation of G_R by repeatedly saturating a non-clique minimal separator will result in a proper interval graph $G_{R''}$ and a corresponding relation R'' such that $\mathcal{L}(R'')$ is a maximal chain of $\mathcal{L}(R)$.*

Proof: This follows directly from Theorem 4.4, as the process of repeatedly removing all concepts not comparable to a concept taken on some maximal chain will result in this chain. □

Property 4.9 *There is a one–to–one correspondence between minimal triangulations of G_R and maximal chains of $\mathcal{L}(R)$.*

Proof: This follows from Properties 4.7 and 4.8, as a proper interval graph is a triangulated graph and as a minimal triangulation $H = (V, E+F)$ of a given graph $G = (V, E)$ is uniquely characterized by the set of minimal separators of H. $\square$

Remark 4.10 A maximal chain of the concept lattice of context $(\mathcal{P}, \mathcal{O}, R)$ has less than $\min(|\mathcal{P}| + |\mathcal{O}|)$ elements and can be obtained in less then n steps, according to Property 4.7. Since each time a minimal separator is saturated the number of concepts decreases, the process of saturating a minimal separator, described by Theorem 4.4, can be repeated as many times as necessary, and can always result in a polynomial-sized sublattice. This may be very useful when the concept lattice is exponentially large, because it allows the user to examine only a part of it.

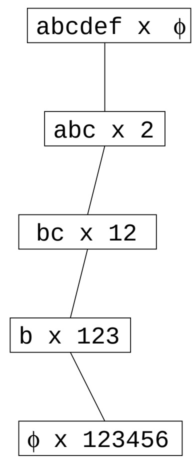


Figure 8. Lattice obtained by computing a minimal triangulation of graph G_R of Figure 3.

5 Using minimal separators to decompose a binary relation and its lattice

In [41], Tarjan introduced the decomposition by clique separators of a graph. This process is defined by repeatedly copying some clique separator S into each of the components it defines. This decomposition is proved to be unique and optimal when only clique minimal separators are used ([26]), and can be described by the following general decomposition step:

Clique Minimal Separator Decomposition Step 5.1 *Let G be a graph, let S be a clique minimal separator of G, defining components $C_1, C_2, ...C_k$. Replace G with $G_1 = G(C_1 \cup N(C_1))$, $G_2 = G(C_2 \cup N(C_2))$,... and G_k =*

$G(C_k \cup N(C_k))$.

This decomposition has the remarkable property that it distributes the minimal separators into the subgraphs it defines.

Property 5.2 *([6]) Let G be a graph, let S be a clique minimal separator of G, let $\mathcal{S}(G)$ be the set of minimal separators of G. After a decomposition step of G by S, the elements of $\mathcal{S}(G) - \{S\}$ are partitioned into the subgraphs obtained.*

In the case of co-bipartite graphs, the clique minimal separator decomposition process is considerably simplified by the fact that each minimal separator defines only two connected components: Decomposition Step 5.1 on clique minimal separator S, defining components $A \subset \mathcal{P}$ and $B \subset \mathcal{O}$, would yield subgraphs $G_1 = G_R(A \cup N(A))$ and $G_2 = G_R(B \cup N(B))$. Since by Characterization 3.8, $N(A) = N(B) = S$, $G_1 = G_R(A \cup S)$ and $G_2 = G_R(B \cup S)$. Moreover, since S is a clique, the vertices of $S \cap \mathcal{P}$ are universal in graph $G_R(A \cup S)$ and according to Property 2.7, they convey no information on minimal separators and can be removed from the graph. The vertices of $S \cap \mathcal{O}$ are likewise universal in $G_R(B \cup S)$, and can be removed. For a co-bipartite graph G_R derived from a binary relation R, we will thus define a simplified decomposition step, which replaces G_R with $G_1 = G_R(A \cup (S \cap \mathcal{O}))$ and $G_2 = G_R(B \cup (S \cap \mathcal{P}))$.

As a consequence of Property 5.2, computing the set of minimal separators of the original underlying graph G_R can be done separately on the smaller subgraphs defined by a decomposition step by a clique minimal separator: T_1 will be a minimal separator of G_1 iff $T_1 \cup (S \cap \mathcal{P})$ is a minimal separator of G_R, T_2 will be a minimal separator of G_2 iff $T_2 \cup (S \cap \mathcal{O})$ is a minimal separator of G_R. Thus the concepts of R can be computed separately on the sub-relations defined. Moreover, it is clear that G_1 contains all the minimal separators representing a concept which is an ancestor of $A \times B$, and that G_2 contains all the minimal separators representing a concept which is a descendant of $A \times B$.

In a co-bipartite graph, the presence of a clique minimal separator can be tested for in linear time, and the decomposition can be computed in the same time ([28]). However, in general, there may not be any clique minimal separator in G_R. We can combine the discussion above with the results from Section 4 and artificially saturate a non-clique minimal separator, and then go on to decompose the graph.

One of the remarkable property of co-bipartite graphs is that the edges added when saturating a minimal separator are not copied into any of the subgraphs defined by the above decomposition, as edges would be added between a vertex of $S \cap \mathcal{O}$ and a vertex of $S \cap \mathcal{P}$. Thus the decomposition step is the same whether or not the clique minimal separator used to decompose the graph is "natural" or "artificial".

We will thus define the following decomposition steps, which can use any minimal separator, whether it is a clique or not:

Co-bipartite Graph Decomposition Step 5.3 *Let G_R be the underlying graph of context $(\mathcal{P}, \mathcal{O}, R)$, let S be a minimal separator of G_R, defining components $A \subset \mathcal{P}$ and $B \subset \mathcal{O}$. Replace G_R with $G_1 = G_R(A \cup (S \cap \mathcal{O})) = G_R(A \cup (O - B))$ and $G_2 = G_R(B \cup (S \cap \mathcal{P})) = G_R(B \cup (P - A))$.*

From Decomposition Step 5.3, we can derive a corresponding relation decomposition.

Relation Decomposition Step 5.4 *Let G_R be the underlying graph of context $(\mathcal{P}, \mathcal{O}, R)$, let $\mathcal{L}(R)$ be the associated concept lattice, let S be a minimal separator of G_R, defining components $A \subset \mathcal{P}$ and $B \subset \mathcal{O}$. Then R can be decomposed into two sub-relations $R_1 = R(A, (O - B))$ and $R_2 = R((P - A), B)$ such that:*

*(1) a concept $X \times Y$ of R is an **ancestor** of concept $A \times B$ in $\mathcal{L}(R)$ iff $X \times (Y - B)$ is a concept of relation R_1. The corresponding sublattice of $\mathcal{L}(R)$, of which $A \times B$ is the top, contains exactly the concepts, the intent of which is a subset of A; it also contains exactly the concepts, the extent of which will be a superset of B.*
*(2) a concept $X \times Y$ of R is a **descendant** of concept $A \times B$ in $\mathcal{L}(R)$ iff $(X - A) \times Y$ is a concept of relation R_2. The corresponding sublattice of $\mathcal{L}(R)$, of which $A \times B$ is the bottom, contains exactly the concepts, the extent of which is a subset of B; it also contains exactly the concepts, the intent of which will be a superset of A.*

Chances are the resulting sub-relations will be much smaller than the original one, and thus the queries on them much less costly.

This process enables us to efficiently answer the following type of query:
"If we have a set of properties X, (for example a set of symptoms in a medical database), which sub-relation should we work on in order to define only the concepts which contain all the properties included in X?"
To do this, we simply:
– compute the smallest concept, the intention of which contains X, and
– extract the sub-relation corresponding to the descendants of this concept.

Example 5.5 *Let us use minimal separator $S = \{a, d, e, f, 3, 4, 5, 6\}$ of Example 3.9. The corresponding lattice is given in Figure 1 in Section 2. S defines components $A = \{b, c\}$ and $B = \{1, 2\}$, thus representing concept $bc \times 12$. $S \cap \mathcal{O} = \{3, 4, 5, 6\}$ and $S \cap \mathcal{P} = \{a, d, e, f\}$.*

A decomposition step using S will yield $G_1 = G_R(C_1 \cup (S \cap \mathcal{O})) = G_{R'}(\{b, c, 3, 4, 5, 6\})$ and $G_2 = G_R(C_2 \cup (S \cap \mathcal{P})) = G_{R'}(\{a, d, e, f, 1, 2\})$, as illustrated by Figure 9 where edges are omitted in cliques $\mathcal{P}$ and $\mathcal{O}$. The initial relation R and its corresponding sub-relations R_1 and R_2 obtained are given in Figure 10.

With a linear-time pass of G_1 it will become clear that vertices 4 and 6 have also become universal and can be removed. Figure 11 shows the very restricted graph G'_1 finally obtained. The minimal separators of G'_1 are $\{b, 3\}$ and $\{c, 5\}$, corresponding to concept $b \times 3$ and $c \times 5$. In the global graph, putting component $C_2 = \{1, 2\}$ back in will yield at no extra cost concepts $b \times 123$ and $c \times 125$ of the original lattice. These are precisely the predecessors of $bc \times 12$.

In G_2, vertex f has become universal, and a linear-time pass will show that vertices d and e now share the same neighborhood, and can be contracted without loss of information on the minimal separators of the graph.

The resulting graph G'_2 is also restricted to four vertices, and is shown in Figure 11. Its minimal separators are represented by $a \times 2$ and $de \times 1$, which, once we have put $C_1 = \{b, c\}$ back in, defines the concepts $abc \times 2$ and $bcde \times 1$, which are the successors of $bc \times 12$.

The corresponding lattice decomposition is illustrated in Figure 12.

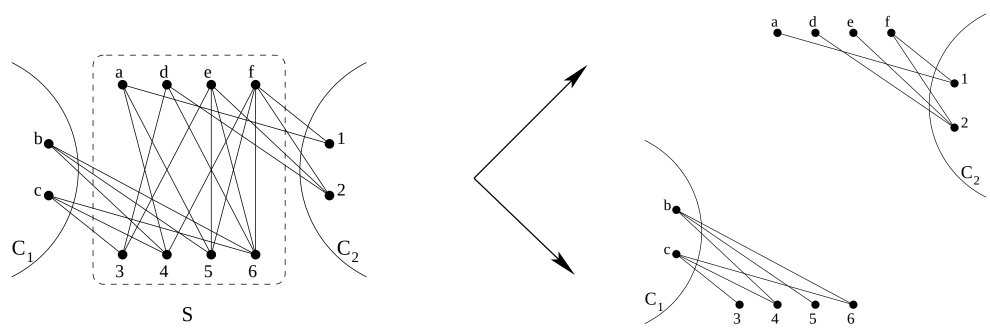


Figure 9. Graphs G_1 and G_2 from decomposition of Example 5.5 (the internal edges of $\mathcal{P}$ and $\mathcal{O}$ are omitted).

6 Computing the cover of an element of the lattice

We will now use the classical properties of graphs to characterize the concepts which constitute the cover of an element of the lattice.

We will need to define the concept of **domination** in a graph:

	a	b	c	d	e	f
1		×	×	×	×	
2	×	×	×			
3	×	×				×
4				×	×	
5			×	×		
6	×					

	a	d	e	f
1		×	×	
2	×			

	b	c
3	×	
4		
5		×
6		

Figure 10. Relations R_1 and R_2 from decomposition of Example 5.5.

Figure 11. The very restricted graphs G'_1 and G'_2 obtained after a decomposition step by minimal separator on G_R in Example 5.5.

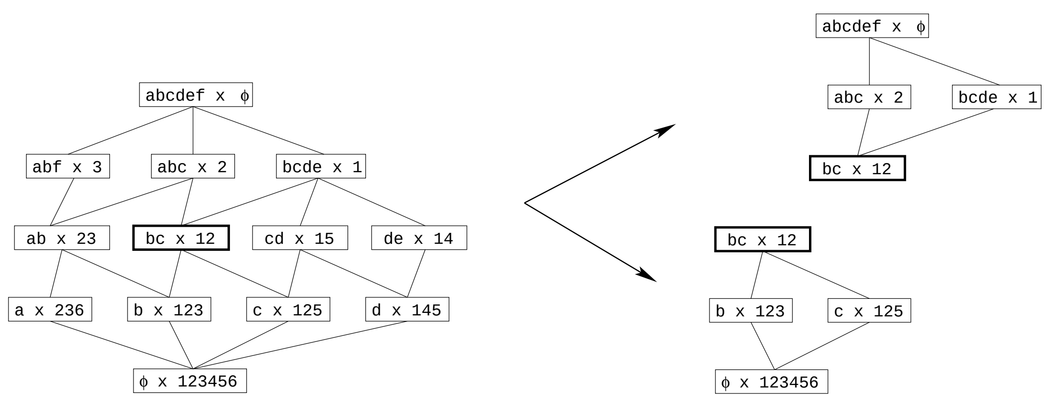


Figure 12. Lattices obtained by decomposition step of Example 5.5.

Definition 6.1 *A vertex x is said to be **dominating** (or strongly dominating) in graph G if there is some vertex y such that $N(y) \subset N(x)$. We will say that a maximal clique module X is dominating if there is some vertex $y \in N(X) - X$ such that $\forall x \in X, N(y) \subset N(x)$. Conversely we will say that a vertex or maximal clique module is **non-dominating** if it is not dominating.*

Property 6.2 *Let $G = (V, E)$ be a co–bipartite graph, let $X \subset V$ be a maximal clique module of G; then X is a moplex iff X is non-dominating.*

Proof: Let $G = (V, E)$ be a co–bipartite graph and $X \subset V$ a maximal clique module of G.

(1) Suppose X is a moplex; then $N(X)$ is a minimal separator inducing (by Characterization 3.5) a second connected component, the neighborhood of which is $N(X)$; thus X is non-dominating.
(2) Suppose X is non-dominating; then $Y = V - (N(X) \cup X)$ has the same neighborhood as X and then $N(X)$ is a minimal separator; thus X is a moplex.

□

Property 6.3 *Let G_R be the underlying graph of context $(\mathcal{P}, \mathcal{O}, R)$, let $X \subset \mathcal{P}$ be a maximal clique module of G_R; then $N(X)$ represents an atom, with intent X, iff X is non-dominating in G_R.*

Proof: As we previously said, we consider only relations without any line of zeroes of or ones. Thus, we use Property 6.2 to reformulate Property 3.13. □

We can use Property 6.3 and the results from the previous section to compute the cover of a given element $A \times B$, by decomposing the lattice and thus obtaining a sublattice of which $A \times B$ is the bottom element.

Theorem 6.4 *A concept $A' \times B'$ covers a concept $A \times B$ iff in $G_2 = G(B \cup (\mathcal{P} - A))$ there is some non-dominating maximal clique module X such that $A' = X + A$.*

For complexity considerations, we need to remark that a maximal clique module X of minimum degree is non-dominating, and that finding the set of vertices which dominate a given maximal clique module X can be done in linear time by checking for universal vertices in $N(X)$.

Our strategy for finding the set of non-dominating maximal clique modules of G_R is the following:

(1) Compute in linear time all the maximal clique modules of G_R and contract them.
(2) Choose a vertex x of minimum degree in the resulting graph.
(3) Compute in linear time the set of vertices which dominate x.
(4) Remove x and the vertices which dominate x from the graph and go back to Step 2.

This requires $O(m)$ time per non-dominated maximal clique module computed.

In Figure 3, the set of maximal clique modules is exactly the set of vertices. Vertices e and f are dominating (e dominates d and f dominates both a and b). $N(a) = \{b, c, d, e, f, 1, 4, 5\}$, $N(b) = \{a, c, d, e, f, 4, 5, 6\}$, $N(c) = \{a, b, d, e, f, 3, 4, 6\}$, and $N(d) = \{a, b, c, e, f, 2, 3, 6\}$, which defines the atoms of $\mathcal{L}(R)$ as: $a \times 236, b \times 123, c \times 125$ and $d \times 145$.

7 Generating the concepts

Recent work has been done on the efficient generation of the concepts defined by a binary relation. One may want to generate and store all the concepts ([30]), or simply encounter each at least once, without storing them ([15]), or one may want to compute the concepts along with their structure, i.e. the arcs of the Hasse diagram of the lattice ([12]).

In parallel, recent work has been done to generate all the minimal separators or all the minimal xy-separators of a graph ([40,24,8,38]).

As an illustration of the use that can be made of our new paradigm, we will show how we can easily match the current best worst time complexities for concept generation using graph results.

When generating and storing the concepts, the current best complexity is held by [30], and is $O(n^2)$ per concept.

Let us use our underlying graph G_R as described in Section 3, and add two simplicial vertices x and y, such that x is a neighbor of all vertices of $\mathcal{P}$, y of all vertices of $\mathcal{O}$. It is easy to see that the set of minimal separators of this new graph is exactly $\{\mathcal{P}\} \cup \{\mathcal{O}\} \cup \mathcal{S}(G_R)$, where $\mathcal{S}(G_R)$ is the set of minimal separators of G_R. Using the results from [38], who claim a complexity of $O(n^2)$ time per minimal xy-separator to generate and store them, we can easily generate and store all the concepts of R in $O(n^2)$ time per concept, noting that [38] claims a better space complexity than [30].

When generating the concepts without storing them, the current best-time complexity is held by Ganter [15], and is $O(|\mathcal{P}^2||\mathcal{O}|)$, i.e. $O(n^3)$.

For this, we will use the results from Section 6 to recursively compute the cover of each element in a depth-first fashion. Since the lattice is of height at most n, such a DFS will require only polynomial space. By the results from Section 6, each element will be generated by its predecessor in linear ($0(m)$) time. A given element will be generated as many times as its number of predecessors, which is at most $|\mathcal{P}|$, as a depth-first traversal easily enables to know whether an encountered element has already been processed. Since $m < |\mathcal{P}|.|\mathcal{O}|$, each element will be generated in $O(|\mathcal{P}^2||\mathcal{O}|)$, which matches the complexity of [15], noting that we generated the Hasse diagram, whereas [15] does not.

Note that our more recent work ([39,9]) uses this process with an adequate data structure maintained which enables us to obtain a better complexity of $O(m)$ per generated concept, plus $O(nm)$ per maximal chain of the lattice traversed by the recursive deapth-first algorithm and the corresponding spanning tree.

8 Conclusion

Though specific problems such as minimizing the number of times a database is accessed remain to be translated in terms of graph separators, we have presented a new approach to answering queries on the concept lattice of a binary relation, which uses a rapidly growing toolbox: the theory of minimal separation in undirected graphs.

We can expect that this approach will create a bridge between the two fields of concept lattice theory and undirected graph theory, and yield new results in both fields.

As noted by one of our referees, which we thank for these remarks, there is a one-to-one correspondence between the maximal chains of the lattice (corresponding to Guttman scales) and the maximal sub-Ferrers relations, which should be investigated in view of our results. It would also be interesting to examine the relationships between concepts, minimal separators, maximal bicliques and minimal hypergraph traversals, which are known to be different facets of the same object.

Moreover, we feel that since the minimal separators of a graph seem to describe the structure of the graph, we have contributed to show a strong semantic aspect behind the concepts defined by a binary relation.

Several open questions arise from the issues discussed in this paper.

It is not known whether the set of non-dominant vertices can be computed in less than linear time per vertex, but improving this would also improve the complexity of the algorithmic process described in Section 7.

Likewise, efficiently computing the set of minimal separators which cross a given minimal separator S would result in a better generation algorithm for concepts.

We have illustrated the use of clique minimal separator decomposition, but other minimal separator-preserving decompositions would directly yield decompositions of a binary relation and of the associated lattice. Conversely, other known decompositions of binary relations might lead to new hole and anti-hole preserving graph decompositions, an important problem in the context of perfect graphs.

Finally, it would be interesting to characterize the binary relations which define a polynomial number of concepts, or the graphs which have a polynomial number of minimal separators; this might help the users of databases to maintain manageable binary relations.

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